A Dynamic Programming Approach to Optimal Planning for Vehicles with Trailers

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***Abstract*— In this paper we deal with the optimal feedback synthesis problem for robotic vehicles with trailers which can be modeled by differential equations in chained-form. With respect to classical methods for numerical evolution of optimal feedback synthesis via Dynamic Programming which are based on both input and state discretization, our method exploits the lattice structure naturally imposed on the reachable set by input quantization. A generalized Dijkstra algorithm can be used to obtain optimal feedback control laws, for chained-form vehicles with n-trailers, in an effective way.**

I. Introduction

A remarkable robotic problem consists in computing motion strategies that bring a system from an initial configuration to a desired final configuration, requiring specific performance to the control policy. Typically this problem is solved by decoupling the task into a path– planning and a path–following sub–tasks.

In our previous work [1], the path planning task has been treated for chained-form systems. In particular, a solution algorithm based on non-standard optimization techniques to the optimal steering problem has been proposed. Convergence to the optimal solution in finite time has been demonstrated. Therefore, the optimal control problem in open–loop has been solved, giving satisfactory results.

In this paper, the same robotic motion problem has been investigated to find feedback motion strategies for chained-form systems. With the proposed approach the two previous sub–tasks are solved at the same time.

Several robotic systems such as mobile wheeled robots and satellites are continuous-time driftless nonholonomic systems that can be converted in the chained-form has it has been shown in [2]. Since then, the chained-form has been extensively been used in the automatic control literature for modeling and controlling several robotic systems ([3], [4], [5], [6], [7], [8], [9]). While many steering methods for chained form systems have been provided in the literature, optimal control for these systems is still a completely open problem.

Under quantization, this class of systems are characterized to have a lattice structure as reachable set, [10], [11], and this property is used in this paper to determine optimal feedback strategies applying Dynamic Programming. In particular, we describe how to apply the Dynamic Programming approach to construct the optimal cost function and the relative motion strategies on the reachable lattice.

Dynamic Programming is a powerful optimization procedure whose drawback is the computational efforts to obtain numerical solutions in case of continuous-state space problems [12], [13], [14]. Hence, the state-space is usually discretized and the discretization mesh affects the accuracy of the approximation to the continuous problem. Furthermore, during the resolution procedure interpolations of values of the value function at points other than state-space grid points may be needed. Hence, optimal solutions depend on the grid resolution chosen on the state-space, on the control discretization and on the interpolation techniques that have been applied [15].

On the contrary, in chained-form systems the lattice structure depends only on the quantized control inputs [11] while no state-space discretization is needed. Furthermore, since all reachable points belong to the lattice (state-space grid) no interpolation is required during the optimal resolution procedure.

The obtained results in computing optimal quantized feedback strategies for chained–form systems can be applied to approximate a solution to the optimal feedback problem for continuous nonlinear systems for which there exist encoding schemes by which the systems can be written in chained-form, [16], [11]. In this paper the particular application to tractors with n trailers is considered. Indeed, such systems can be converted in chained form, as has been shown in [3] by Sørdalen with a constructive method.

The paper is organized as follows. The dynamic programming approach to optimal control is introduced in section II. In section III the Dynamic Programming application to chained-form systems and a resolution algorithm are described. The applications of the proposed approach to wheeled vehicles with trailers is reported with experimental results in section IV.

II. The Classical Dynamic Programming

Consider a discrete-time stationary dynamical system

$$x_{t+1} = f(x_t, u_t), \quad u \in W,$$

where t denotes a time stage and f is the *state transition function*. Given an initial configuration $x_{start} = x_1$, let $u_1, \ldots, u_K$ be an input sequence lead into a state trajectory

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or a path, $x_1, \ldots, x_{K+1}$. We define a *trajectory cost* function

$$L(x_1, \ldots, x_{K+1}, u_1, \ldots, u_K) = \sum_{k=1}^{K} l(x_k, u_k) + l_{K+1}(x_{K+1}), \tag{1}$$

where $l(x_k, u_k)$ is the cost related to the state x_k and to the control u_k, and $l_{K+1}(x_{K+1})$ yields the cost of the final state. Hence, $L(x_1, \ldots, x_{K+1}, u_1, \ldots, u_K)$ is the cost of the trajectory produced by the control $u_1, \ldots, u_K$, starting at state x_{start}.

Given a desired configuration $x_{goal} = x_{K+1}$, the optimal feedback control problem consists in finding a feedback control sequence that achieves the goal and minimizes the trajectory cost function.

The resolution of this type of problems is the goal of the celebrated Dynamic Programming approach, proposed by Bellman [17] and later extended to a complete mathematical and computational theory by Larson [14], [18] and Bertsekas [13].

Solving this problem (see [15]) by Dynamic Programming consists in constructing a representation of the feedback strategy ($\gamma : \mathcal{X} \mapsto U$ where $\mathcal{X}$ is the state space and U is the input space) in terms of a navigation function that we indicate as the *optimal cost function*. This function is defined as $L^* : \mathcal{X} \mapsto [0, \infty]$ and associates at each state, the cost given by the trajectory cost function (1) under the execution of the optimal trajectory from the state to the goal configuration. If the goal configuration can not be reached, the value of L^* is infinity.

Numerically, the function L^* is computed as limit of a finite sequence of stage-dependent optimal cost functions, accordingly with the fact that for all points taken in consideration, the goal is reached in a finite number of stages.

Typically, in Dynamic Programming, the procedure is backward from the goal to the initial state and consists in considering k-trajectories for all $k \in \mathbb{N}$, where a k-trajectory is a trajectory obtained from the goal state applying k (inverse) control inputs. Since the number of stages to reach the initial state is finite, the procedure stops as soon as the number of stages is higher than the minimum number of stages necessary to check an optimality criterion [15], [14]. Hence, starting from the goal state at stage 0, an optimal cost function L_k^* can be defined for each stage k as follows

$$L_k^*(x) = \min_{u_k} \left\{ l(x, u_k) + L_{k-1}^*(f(x, u_k)), \right\} \tag{2}$$

where $L_0^*(x_{goal}) = 0$ and the value $L_k^*(x)$ yields the optimal cost for the path from state $x_{goal} = x_{K+1}$ to x in k stages. If we consider two different stages k_1 and k_2, with $k_1 < k_2$, $L_{k_1}^*(x) \geq L_{k_2}^*(x)$ is satisfied.

In case of optimal paths of unspecified length (in terms of stages) the approach is similar to the one described [15]. A "null" action u_N is introduced such that $\forall i \geq k$, $u_i = u_N$, $x_i = x_k$ and $l(x_i, u_N) = 0$. Furthermore, notice that formula 2 can be iterated for infinite times. If the function $l(x, u)$ is non negative there exist a *stationary* $\hat{k}$ such that $L_{\hat{k}+1}^*(x) = L_{\hat{k}}^*(x) \forall x \in \mathcal{X}$. Under these assumptions the optimal cost function can be defined as follows

$$L^*(x) = \min_{u} \left\{ l(x, u) + L^*(f(x, u)) \right\}. \tag{3}$$

Also in this case the optimal cost function is evaluated starting from the final state x_{goal} and is obtained recursively proceeding with a backward approach.

In general, the optimal cost function L^* is numerically computed over a grid of points, then approximated on the other points by interpolation. In the particular case of chained-form systems, interpolation is not needed because all reachable points belong to the lattice and for each intermediate point the function L^* is computed. A fundamental property of dynamic programming approach is that the optimal solution depends on the choice of the grid points and on the control discretization. In [19], [14] the convergence of the approximated values to the optimal one while increasing the resolution of the grid has been shown.

The optimal cost function L^*, computed iteratively as above, is used to encode the feedback strategy

$$\gamma(x_t) = \text{argmin}_{u_t \in U} L^*(x_{t+1}), \tag{4}$$

that is the input $\gamma(x_t)$ is obtained as a control $u_t \in U$ that yields the minimum value for the optimal cost function on the next state (4).

The Dynamic Programming approach will be applied in the following for our purposes. In particular, it will be applied and modified for a problem with a fiber structure such as the optimal steering problem for chained-form systems with quantized inputs.

III. Optimal Feedback on Lattices through Dynamic Programming

An algorithm that computes optimal steering strategies for quantized chained-form systems has been proposed in [1]. We are now interested in computing optimal feedback strategies on the reachable lattices generated by quantized chained-form systems. For the reader convenience, some notation and fundamental results reported in [11], [1], [20] are now briefly described.

A. Main properties of chained form systems

Chained-form systems, introduced by Sastry and Murray in [2] as a canonical form for some continuous-time, driftless nonholonomic systems, are described by the ordinary differential equations

$$\begin{cases} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_2 u_1, \\ \vdots &= \vdots \\ \dot{x}_n &= x_{n-1} u_1. \end{cases} \tag{5}$$

Consider the case where system inputs, rather than being allowed to change continuously in time, are bound to switch among a finite set of different levels at given switching times, which are multiples of a given time interval. Assuming such sampling interval to be of unit length, an exactly sampled model of chained-form systems can be easily obtained in discrete time from (5) by integration as

$$\left\{\begin{array}{lcl} x_1^+ & = & x_1 + u_1, \\ x_2^+ & = & x_2 + u_2, \\ x_3^+ & = & x_3 + x_2 u_1 + \frac{1}{2} u_1 u_2, \\ \vdots & = & \vdots \\ x_n^+ & = & x_n + \sum_{j=1}^{n-2} x_{n-j} \frac{u_1^j}{j!} + u_1^{n-2} u_2 \frac{1}{(n-1)!}. \end{array}\right. \quad (6)$$

We will assume that inputs $u = (u_1, u_2)$ can take values within a state-independent set of input symbols U, which is symmetric (i.e., if $u \in U$, then also $\bar{u} = -u \in U$). The set Ω of admissible control words (i.e. strings of admissible input symbols) is endowed with a composition law given by concatenation of strings. Because of the symmetry of U, every element $\omega \in \Omega$ has an inverse $\omega^{-1} \in \Omega$, simply defined as $(u_1 u_2 \cdots u_q)^{-1} = -u_q \cdots - u_2 - u_1, \pm u_i \in U, \forall i$.

In the state manifold of chained-form systems (5, 6) it is customary to distinguish a *base* subsystem, consisting of the first two state variables (x_1, x_2), and a *fiber* subsystem with coordinates $(x_3, \ldots, x_n)$. Observe that the restriction of chained-form systems to the base variables is linear, and indeed trivial to control. On the other hand, the difficulty in controlling fiber variables increases with the dimension of the state space.

Regarding the reachability properties of the two subsystem the following holds ([10]): if the controls set U is rational and quantized, the reachability structure of a chained form discrete-time system is completely described by a lattice in the state space (the cartesian product of the base and fiber lattices).

Furthermore, the lattice structure, which plays a central role in our approach in solving the optimal steering problem, can be completely described by a finite number of generators, whose evaluation can be done in polynomial time with respect to the state space dimension and the number of control symbols in U ([11]).

From now on, we focus on the fiber subsystem of (6) with quantized, rational and symmetric control set U for which the reachable set has a lattice structure. Denote this lattice structure as ψ with state space $\mathcal{X}$ in which each state is a vector that represents a fiber reachable displacement. The problem consists in designing an optimal feedback motion strategy that steers the fiber subsystem from a desired final configuration x_{goal} to x_{start} with the assumption that these configurations belong to the lattice, see figure 1. Otherwise we consider the best approximations of these points on the lattice and we solve the feedback problem with an error related to the lattice mesh.

Since the lattice is obviously invariant by translations, x_{goal} can be considered to be 0 translating the relative x_{start} in $\hat{x}_{start} = x_{start} - x_{goal}$. Hence, the task is to determine the optimal feedback from the origin to the configuration $\hat{x}_{start}$.

Let $\mathcal{I} = \{g_1, \ldots, g_l\}$ denotes the displacements on the fiber that generate the lattice, $\mathcal{W} = \{\omega_1^1, \ldots, \omega_{j_1}^1, \omega_1^2, \ldots, \omega_{j_2}^2, \ldots, \omega_1^l, \ldots, \omega_{j_l}^l\}$ a set of cyclic controls on the base that generate the above displacements,

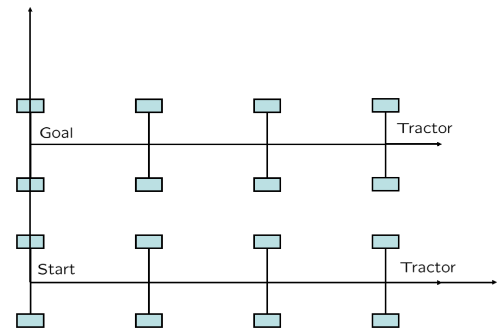

Fig. 1. Example of steering of a 3-Trailer system on the fiber

[22], [11]: in particular the motion g_i can be realized by any cyclic control in $\{\omega_1^i, \ldots, \omega_{j_i}^i\}$. Let Ω be the set of the finite control sequences whose control words belong to the set $\mathcal{W}$: we have that any displacement on the fiber lattice is obtained by a word sequence $\tilde{\omega} \in \Omega$.

Given a control word $\tilde{\omega}$, define the function $\Gamma : \Omega \mapsto [0, \infty]$ that yields the cost of the sequence. This value is computed as the number of symbols that compose the word (minimum-time optimization) or by a weighted sum of the number of symbols that appear in the control sequence. By results in [11], the cyclic controls in $\mathcal{W}$ are fiber generators of lowest cost.

Furthermore, composing cyclic controls different number of cancellations may occur. For example, if $\omega_i = u_1 u_2 u_3 u_4$ and $\omega_j = -u_4 u_5 - u_2 - u_1$, in a minimum time problem we have $\Gamma(\omega_i) = 4$ and $\Gamma(\omega_j) = 4$. However, the concatenation of ω_i with ω_j leads, by cancellations, to the control word $\omega = u_1 u_2 u_3 u_5 - u_2 - u_1$, so that $\Gamma(\omega) = 6 < \Gamma(\omega_i) + \Gamma(\omega_j)$. Obviously, cancellations are crucial in minimizing unnecessary maneuvers in the steering problem, and motivate the necessity of introducing a generalized lattice structure that takes into account a larger amount information for each state. A possible generalized lattice structured is described in next subsection.

B. The Generalized Lattice

Let $\tilde{X} = \mathcal{X} \times \mathcal{W} \times \mathcal{W}$ be a generalized state space, where $\mathcal{X}$ is the state space of the initial problem and $\mathcal{W}$ the control set of lattice generators. We consider a subset $\Psi \subset \tilde{X}$ whose points are defined as $\tilde{x} \in \Psi, \tilde{x} = (x, \omega, \omega_l)$ where x is a point of the reachable lattice ψ in the space $\mathcal{X}$, ω and ω_l are symbols belonging to the set $\mathcal{W}$. The subset Ψ has a lattice structure inherited from the lattice structure ψ of the first component while the other components belong to finite sets.

Let us introduce an equivalence relation $\sim$ over the elements of Ψ by setting $(x_1, \omega_1, \omega_{l1}) \sim (x_2, \omega_2, \omega_{l2})$ iff $x_1 = x_2$. Using this relation, the lattice $\psi \in \mathcal{X}$ can be obtained considering the quotient $\tilde{\Psi} = \Psi / \sim$ with respect to the equivalence relation $\sim$. The equivalence class of state $\xi = (x, \omega, \omega_l)$ will be denoted as $[x]$. Hence, a path from the origin to $\hat{x}_{start}$ on the lattice ψ becomes a path on $\tilde{\Psi}$ between $[0]$ and $[\hat{x}_{start}]$.

The *state transition function* on the generalized lattice Ψ is given by

$$\xi_{t+1} = f_\Psi(\xi_t, \omega) = (x_t + g, \omega, \omega_t), \quad \omega \in \mathcal{W},$$

where $\xi_t = (x_t, \omega_t, \omega_{lt})$ and g is the fiber displacement generated by ω.

Notice that in the state we keep track of both the last and the second-last control used to reach the state. This is necessary in order to use the inverse of the state transition function in the backward approach of Dynamic Programming and for the reconstruction of the optimal feedback control sequence. However, the equivalence class is represented only by the fiber displacement.

C. The optimal cost functions

Optimal cost functions L^* and $\hat{L}^*$ will be now defined for the fiber lattice ψ and the generalized latice Ψ. The relation between these quantities is given by the quotient space $\Psi / \sim$. Indeed, for each equivalence class in $\Psi / \sim$, it is possible to define an optimal cost function on $\Psi / \sim$ as follows

$$\tilde{L}^*([\xi]) = \min_{\zeta \in \Psi, \zeta \in [\xi]} \hat{L}^*(\zeta).$$

Since in any equivalence class we have a finite number of points, this operation is well defined. Besides, considering the equivalence class corresponding to the point x in ψ, we can define the function $L^*(x) = \tilde{L}^*([x])$. It is fundamental to observe that an equivalence class can be visited different times with different values of the optimal cost function.

For simplicity of presentation, we consider the computations of the minimum–time optimal feedback trajectories, where the value of $\Gamma(\omega)$ is the number of symbols in ω (or equivalent its length). The extension to weighted control costs is straightforward.

Since the goal configuration (starting point of the backward approach) is the zero of the fiber space we have that $\tilde{L}^*([0]) = 0$ and $L^*(0) = 0$. Indeed, any points the fiber can be reached with negative costs.

From (3), the optimal cost function $\hat{L}^*(\xi)$ defined on the points $\xi = (x, \tilde{\omega}, \tilde{\omega}_l) \in \Psi$ of the generalized fiber is rewritten as

$$\hat{L}^*(\xi) = \min_{\omega \in \mathcal{W}} \left\{ l(\xi, \omega) + \hat{L}^*(f_\Psi(\xi, \omega)) \right\}. \tag{7}$$

The cost term $l(\xi, \omega)$ can be defined as

$$l(\xi, \omega) = \Gamma(\omega) - canc(\tilde{\omega}, \omega),$$

where $canc(\tilde{\omega}, \omega)$ is the number of symbols cancellations that occur in the composition of $\tilde{\omega}$ and ω (in the example reported in section III the number of cancellations is 2).

In order to have stationary conditions it is necessary to have a non negative cost term. Unfortunately, the cost term may take negative value when the number of cancellations is larger than the length of the applied control. Hence, the sequence of ω and ω^{-1} should not be allowed. This constraints is also justified by the fact that the sequence $\omega\omega^{-1}$ corresponds to a unecessary maneuver for the vehicle motion planning problem. The introduction of this constraint is not sufficient to ensure a non negative cost term. For example, if $\tilde{\omega} = u\,v\; - u\,v\,u$ and $\omega = -u\; - v\,v$ we have $\tilde{\omega}\omega = u\,v\; - u, v$, $canc(\tilde{\omega}, \omega) = 4$ and $l(\xi, \omega) = 3 - 4 = -1$. To avoid this problem, we introduce another generator in $\mathcal{W}$ obtained by the composition of $\tilde{\omega}$ and ω with the relative fiber contribution added in $\mathcal{I}$ and relative length $\Gamma(\tilde{\omega}\omega)$, i.e. in the case above $\Gamma(\tilde{\omega}\omega) = 4$ and the fiber contribution is given by the sum of the two controls contributions. Furthermore, the application of control ω to a state ξ with $\tilde{\omega}$ as second component is no more allowed, see figure 2. This procedure is repeated for each pair of controls in $\mathcal{W}$ whose composition leads to a number of cancellation that is larger than the length of the first control. With this strategy, the cost term applied to the augmented control space $\mathcal{W}$ (still denoted with $\mathcal{W}$ for simplicity of notation) is non negative.

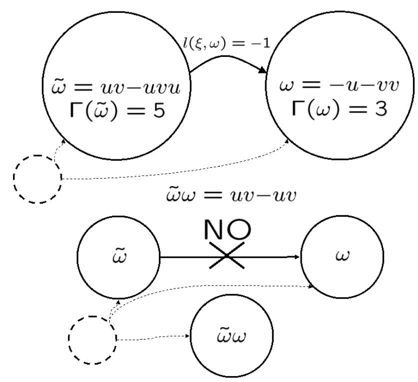

Fig. 2. Sequences concatenation that leads to a large number of cancellation

D. Maximum optimal costs and optimal exploration of fiber sectors

As introduced in section II, applying dynamic programming techniques, the optimal strategy is constructed with a backward approach, starting from final state x_{goal} (hereinafter considered as translated to the origin) to initial state x_{start}.

Each state of the fiber can be associated to a node of a graph. An arc from the node associated to $\xi = (x, \omega, \omega_l)$ to the node associated to $\zeta = (y, \nu, \nu_l)$ exists if $\zeta = f_\Psi(\xi, \nu)$, i.e. $y - x$ is the fiber displacement generated by ν, and $\nu_l = \omega$. Furthermore, such arc exists if the number of cancellations in the composition of ν and ω is smaller than $\Gamma(\nu)$. In particular, ν cannot be the inverse of ω. Finally, the cost $l(\xi, \nu)$ is associated to the arc from ξ to ζ.

Starting from the node associated to the origin of the fiber, a generalized Dijkstra algorithm is applied backward to the obtained graph. Reversing the results obtained with this approach it is possible to find feedback optimal control laws to steer the system from any point of a fiber sector to the origin.

The following remarks provide a quantitative description of the fiber space effectively explored by the Dijkstra algorithm and of the maximum optimal cost that we may expect to find in a sector. Let $r(\tilde{\omega},\omega) = \Gamma(\omega) - canc(\tilde{\omega},\omega)$ for $\tilde{\omega},\, \omega \in \mathcal{W}$, $c = \min_{\tilde{\omega},\,\omega\in\mathcal{W}} r(\tilde{\omega},\omega)$ and $C = \max_{\tilde{\omega},\,\omega\in\mathcal{W}} r(\tilde{\omega},\omega)$ the minimum and maximum cost of the arcs of the graph. Let $\delta = (\delta_1,\, \ldots, \delta_{n-2})$ where $\delta_i = \min_{g\in\mathcal{I}} |g_i|$ and index i represents the i-th fiber component. Equivalently, let $\Delta = (\Delta_1,\, \ldots, \Delta_{n-2})$ where $\Delta_i = \max_{g\in\mathcal{I}} |g_i|$.

Remark 1: The optimal feedback control law to reach a given fiber displacement P has cost $L^*(P) = \tilde{L}^*([P])$ whose upper bound is $U_b = \max_{i=1,\,\ldots,\,n-2}\left(\frac{P_i}{\delta_i}\right) C$.

Remark 2: A fiber sector centered in the origin and with vertex P is formed by points for which the optimal cost is obtained if the Dijkstra algorithm has explored all nodes of cost at most U_b.

Remark 3: Let $\Delta^M = \frac{U_b}{c}\Delta$ and consider the fiber sector M centered in the origin and with a vertex in Δ^M. By exploring with the Dijkstra algorithm all nodes of cost at most U_b no fiber point out of M need to be explored by the algorithm.

Remark 4: Let explore with the Dijkstra algorithm all nodes whose optimal cost is less or equal to $\hat{C}$, and let $\delta^m = \frac{\hat{C}}{C}\delta$. At least for all nodes in the fiber sector m centered in the origin and with a vertex in δ^m the optimal cost has been obtained.

The above propositions allow the introduction in the Dijkstra algorithm of constraints in the exploration of the graph since they determine which node of the graph does not need to be explored. It is important to notice that the application of the Dijkstra algorithm do not require to store in memory the entire graph but only explored nodes.

IV. Application to wheeled vehicles

According to the reachability lattice approach [16], the results obtained on lattices can be extended to reticulability systems. In particular, these results can be applied to systems that are equivalent to chained–form systems, in the sense of coordinate changes and state feedback.

In this section, the optimal steering problem with feedback strategies is applied to a tractor with n trailers. Experimental results of the proposed generalized Dijkstra approach are also reported.

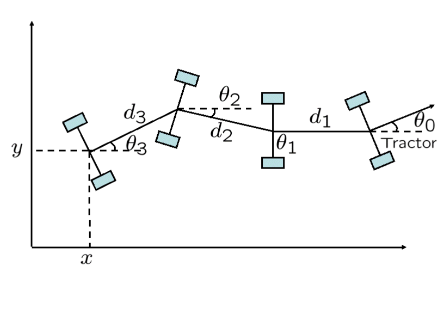

Fig. 3. A 3-Trailer system, in this case $n = 6$.

Referring to figure 3, the kinematic model of a tractor with n trailers is given by

$$\begin{cases} \dot{x} &= \cos\theta_n v_n \\ \dot{y} &= \sin\theta_n v_n \\ \dot{\theta}_n &= \frac{1}{d_n}\sin(\theta_{n-1}-\theta_n)v_{n-1} \\ &\vdots \\ \dot{\theta}_1 &= \frac{1}{d_1}\sin(\theta_0-\theta_1)v_0 \\ \dot{\theta}_0 &= \omega \end{cases} \tag{8}$$

where (x, y) is the absolute position of the center of the axle between the two wheels of the rear-most trailer; θ_i is the orientation angle of trailer i with respect to the x-axis, with $i \in \{1, \ldots, n\}$; θ_0 is the orientation angle of the tractor axle with respect to the x-axis; d_i is the distance from the center of trailer i to the center of trailer $i-1$, $i \in \{2, \ldots, n\}$; d_1 is the distance from the wheels of trailer 1 to the wheels of the tractor while d_i is the distance from the wheels of the $i-1$-th trailer to the wheels of the i-th trailer. The two inputs of the systems are v_0 and ω, the tangential velocity of the car and the angular velocity of the tractor respectively. The tangential velocity v_i, of trailer i, is given by

$$v_i = \cos(\theta_{i-1}-\theta_i)v_{i-1} = \prod_{j=1}^{i}\cos(\theta_{j-1}-\theta_j)v_0,$$

where $i \in \{1, \ldots, n\}$. Incidentally, this model is identical to the model of a four-wheeled car pulling $n-1$ trailers, provided $\theta_0 - \theta_1$ denotes the angle of the front wheels relative to the orientation θ_1 of the rear axle of the four-wheeled car.

System (8) can be converted in chained form, as has been shown in [3] by Sørdalen with a constructive method. As already mentioned, under quantization on the control inputs, this class of systems are characterized to have a lattice structure as reachable set. The described quantized feedback optimal control algorithm can then be applied to the approximate determination of an optimal continuous control for system (8). Computed solutions will provide piece-wise continuous inputs to system (8).

For this application, the considered quantized control set is comprised of three inputs:

$$U = \left\{ \pm\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \pm\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \pm\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$

corresponding respectively to straight motions, rotations about the axle center and the tractor turn motions.

Several simulations have been carried out from the unicycle case $n = 3$ up to the 5-trailer case ($n = 8$). By fixing a maximum optimal cost value (MOC) we report in the following table the CPU time (in seconds), the number of optimal nodes obtained (ON) and the dimension of the minimum fiber sector explored (δ^m).

The computation has been done on a Windows Xp, Pentium 4, 2GHz, 512MB RAM.

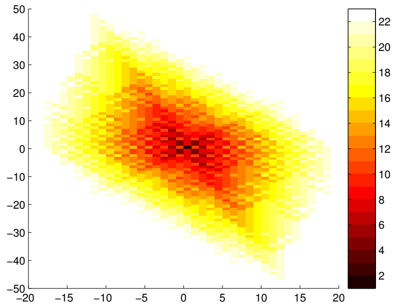

Fig. 4. Nodes with optimal cost less or equal than 20 represented on the fiber space for the 1-Trailer system (a scale of 2 on first component and of 6 on second component have been used)

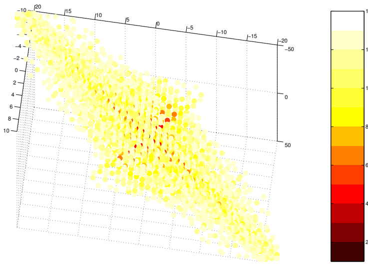

Fig. 5. Nodes with optimal cost less or equal than 11 represented on the fiber space for the 2-Trailer system (a scale of 2 on first component, of 6 on second component and of 24 on third component have been used)

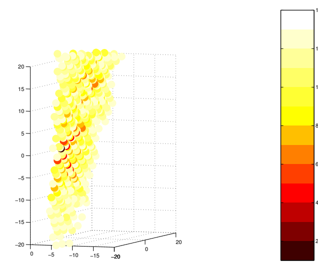

Fig. 6. Zoom of nodes with optimal cost less or equal than 11 for the 2-Trailer system

	MOC	ON	sec	δ^m	Δ^M
U	500	11917	18	83	250
U	1000	23917	89	166	500
1-T	20	28711	133	δ_1^m	$2\Delta_1^M$
1-T	30	79799	1362	$\frac{3}{2}\delta_1^m$	$3\Delta_1^M$
2-T	15	143709	2507	δ_2^m	$3\Delta_2^M$
2-T	20	822206	97599	δ_2^m	$4\Delta_2^M$
3-T	10	5554	1.6	δ_3^m	$10\Delta_3^M$
3-T	12	24621	68	δ_3^m	$12\Delta_3^M$
3-T	15	280423	9059	δ_3^m	$15\Delta_3^M$
4-T	10	5754	2.0	δ_4^m	$10\Delta_4^M$
4-T	12	29324	77	δ_4^m	$12\Delta_4^M$
4-T	15	314275	11850	δ_4^m	$15\Delta_4^M$
5-T	10	6894	4.2	δ_5^m	$10\Delta_5^M$
5-T	12	36152	184	δ_5^m	$12\Delta_5^M$

where $\delta_1^m = \left(2\ \frac{2}{3}\right)^T$, $\Delta_1^M = (10\ 10)^T$, $\delta_2^m = \left(\frac{1}{2}\ \frac{1}{6}\ \frac{1}{24}\right)^T$, $\Delta_2^M = (5\ 10\ 10)^T$, $\delta_3^m = \left(\frac{1}{2}\ \frac{1}{6}\ \frac{1}{24}\ \frac{1}{120}\right)^T$, $\Delta_3^M = (1\ 3\ 5\ 5)^T$, $\delta_4^m = \left(\frac{1}{2}\ \frac{1}{6}\ \frac{1}{24}\ \frac{1}{120}\ \frac{1}{720}\right)^T$, $\Delta_4^M = (1\ 4\ 2\ 12\ 12)^T$, $\delta_5^m = \left(\frac{1}{2}\ \frac{1}{6}\ \frac{1}{24}\ \frac{1}{120}\ \frac{1}{720}\ \frac{1}{5040}\right)^T$, $\Delta_5^M = (1\ 5\ 13\ 22\ 27\ 28)^T$.

In figure 4 the nodes with optimal cost less or equal than 20 for the 1-trailer system have been reported on the fiber space (a scale of 2 on first component and of 6 on second components have been used). The sector delimited by δ_1^m (scaled accordingly) has been visited while no nodes external to the sector delimited by $2\Delta_1^M$ (scaled accordingly) has been explored.

In figures 5 and 6 the nodes with optimal cost less or equal than 11 have been reported on the 3D fiber space for the 2-Trailer system (a scale of 2 on first component, of 6 on second component and of 24 on third component have been used).

V. Conclusion

An algorithm to compute feedback motion strategies for chained-form systems by quantization on the control inputs has been proposed. For this systems class, under quantized rational inputs, the structure of the reachable set is a lattice and this property plays a central role in the definition of the feedback law. Our approach in computing optimal trajectories is inspired by dynamic programming and in particular the feedback strategy is encoded by an optimal cost function that, in robotics, can be seen as the navigation function.

Applying Dynamic Programming directly on the lattice structure we eliminate the discretization on the state-space and the interpolation tasks, usually connected to this optimal control techniques. Furthermore, satisfactory numerical results can be obtained, as shown above.

These results provide a method to compute optimal feedback strategies for a large class of nonholonomic continuous systems that can be converted in chained form by coordinates change and feedback.

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