Optimal exploratory paths for a mobile rover

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Abstract

In this paper, we consider the problem of maximizing the localization accuracy of a mobile vehicle, based on triangulation measurements derived from optical data. The problem is intrinsically nonlinear, as the linear approximation of the system is not observable. This implies that the choice of inputs (i.e., the path followed) may affect the quality of observations made, and ultimately the localization accuracy. We consider the problem of finding the most informative exploratory path of given length for a rover (modeled as a point in the plane) with optical triangulation information.

1 Introduction

One of the main technical difficulties in applying mobile robots to unstructured environments is the problem of localization of the vehicle with respect to the environment, and of constructing a map of the environment itself. The problem is important for instance for a rover exploring an unknown terrain, such as was the case for the Sojourner explorer in the recent NASA mission of Pathfinder on Mars. On the other hand, many everyday applications on Earth call for solutions to the same problem. There is in fact a consolidated trend in industrial AGV systems to move from traditional wire–guided systems to optically guided systems, using laser or camera heads on the vehicle to locate it on the factory floor. The advantage of the latter techniques is apparent, in terms of drastically reducing the cost and rigidity of fixed nets of active or passive magnetic devices placed under the floor, and allowing more variate trajectories to be executed by the AGV's. On the other hand, the technology of optical localization is rather new, and several problems are still encountered, related to both its technological aspects and to methodology to be used in filtering and merging data from different sources. A good treatment for these problems is that of Borenstein [3], and references therein.

In this paper, we deal with the problem of localization and map building for a mobile vehicle endowed with odometric and optical sensors (laser or camera heads). In section 2 we recast the problem as one in nonlinear observability, and results obtained from differential–geometric nonlinear system theory are compared with those resulting from a linearized model, showing how the problem is intrinsically nonlinear. In section 3 we pose, and solve for a simplified rover model, the problem of finding the most informative exploratory path of given length with optical triangulation information.

2 Nonlinear Observability

We consider a system comprised of a mobile vehicle, such as a robotic rover in a planetary exploration mission, which moves in an unknown environment with the aim of localizing itself and the environment features (in the rover case, e.g., rocks and geological formations). The vehicle is endowed with a sensor head such as a radial laser rangefinder or movable camera, whose data are assumed here to have been preprocessed so as to yield a measurement of the azimuth angle in the horizontal plane between the line joining the obstacle features with the head position, and the direction of movement of the vehicle (or any other direction fixed w.r.t. the vehicle). An information on distance of the target from the head is not considered to be available, due to the fact that such measurement is hard to obtain accurately from current laser or camera sensors.

Both the vehicle initial position and orientation, and the obstacle positions, are unknown (or, more generally, known up to some a priori probability distribution). The task is to reconstruct such information from angular measurements.

A model of the system that captures most salient features of the problem, yet lends itself to simple analytical results, is used, which is based on the following assumptions: the vehicle moves on a plane, and object features are represented as points of the plane. Among the features that the sensor head detects in the robot environment, we will distinguish between those belonging to objects with unknown positions (which we shall call *targets*), and those belonging to objects whose absolute position is known, which will be referred to as *markers*.

The vehicle dynamics are supposed to be slow enough to be neglected (dynamics do not add much to the problem structure, while considerably increase formal complexity). The mathematical model of the systems is linear in control, and will be described as

$$\begin{array}{rcl} \dot{x} & = & G(x)u \\ y & = & h(x) \end{array} \qquad (1)$$

where x is an n–dimensional state, u an m-dimensional input, $g(x)$ an n by m matrix whose columns are the input vector fields, and y is a p dimensional output vector.

Generally speaking, the kinematics of the vehicle might be nonholonomic: for instance, NASA's Sojourner could be modeled as a unicycle. In this case, the vehicle's state can be described by the position of the sensor head center, and by an orientation angle. The position in the plane of the N targets, whose reconstruction is part of the problem, can be considered as additional $2N$ states (with trivial dynamics) (see fig.1). Outputs in this examples would be the $M + N$ angles

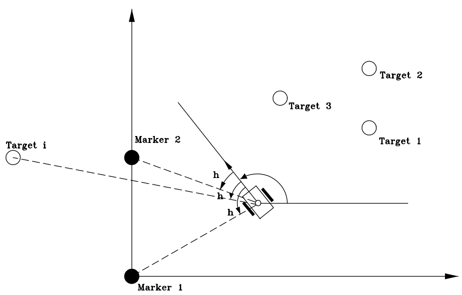

Figure 1: A unicycle in an unknown environment with markers and targets.

formed by the lines through the sensor head and the M markers and N targets, with the rovers fore axis. It can be easily seen in this case that the linear approximation of system (1) is not completely observable if there are targets ($N \neq 0$) or if there are less than three markers ($M < 3$): indeed, the drift term being null, only static measurements are available to the linearized model. On the other hand, it is intuitively clear that triangulation may allow reconstruction of all the problem unknowns, except at most for singular configurations. This can be verified by a nonlinear observability analysis, which has been reported in detail in [2].

It is well known that in nonlinear systems, as opposed to linear ones, observability may depend on inputs. In our rover localization problem, this implies that there may be trajectories that allow reconstruction, as well as others that do not. Two simple examples illustrating this fact are reported in fig.2 (a detailed study is reported in [2])

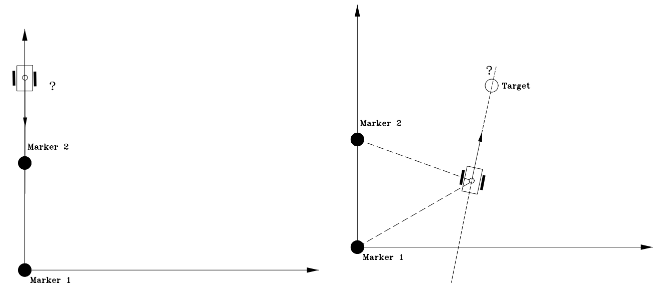

Figure 2: A vehicle triangulating with two markers cannot localize itself if and only if the trajectory aims at the two markers; it cannot localize a target if it aims at the target directly.

3 Optimal exploratory paths: Problem Statement

In the previous section we have seen that, depending on the trajectories followed by the rover, its localization may become impossible. Naturally, it is to be expected that not only the existence of unobservable states is affected by trajectories, but also possible quantitative measures of information collected along the trajectory.

One such quantitative measure can be defined as follows. Consider the output function $y(t) = h(x(t))$ as a function of the initial conditions x_o and of the input functions $\mathbf{u} \in U$, with U a suitable functional space, and denote this as $y(x_o, u, t)$. Let x_o^o and x_o' denote two different initial conditions, with $\|x_o^o - x_o'\| < \epsilon$, and write

$$y(x_o', u, t) - y(x_o^o, u, t) = \left.\frac{\partial y}{\partial x_o}\right|_{x_o = x_o^o} (x_o' - x_o^o) + O^2(\epsilon)$$

In order to distinguish between x_o^o and x_o' based on the difference in outputs, premultiply both sides by $\left.\frac{\partial y}{\partial x_o}\right|_{x_o = x_o^o}^T$ (denoted $\frac{\partial y}{\partial x_o}$ for short) and integrate from time 0 to T to get

$$\int_0^T \frac{\partial y}{\partial x_o}^T (y(x_o', u, t) - y(x_o^o, u, t))\, dt + O^2(\epsilon)$$

$$= \left(\int_0^T \left(\frac{\partial y}{\partial x_o}^T \frac{\partial y}{\partial x_o} \right) dt \right) (\mathbf{x}_o' - \mathbf{x}_o^o)$$

This equation has the form of a linear system $b + \delta = Fx$, where the known vector b comes from measurement outputs, the perturbation term δ comes from approximations errors (and possibly from measurement noise), and matrix F depends on inputs. It can be shown that invertibility of F is tantamount to observability of the system if $h(x)$ is analytic. Also, in order to have the least propagation of perturbations δ in the solution x, it is well known (see e.g. Bicchi and Canepa [1]) that some norm of the inverse of F should be minimized. Notice that such criteria does not reflect any particular choice in the estimator or filter adopted in the actual localization procedure, rather it is intrinsic to the reconstructibility of the state from the given trajectory. A characterization of the criterion in terms of the Fisher information matrix and the Cramer–Rao bounds associated to the problem was presented by Piloni and Bicchi [7].

In the rest of this paper we will consider the problem of maximizing a quantitative measure of observability embodied in the minimum eigenvalue of F. However, in applications such as the exploration of a planet's soil by a robotic rover, the system is also confronted with limitations in autonomy of motion, e.g. in the total length of the path the vehicle can track in one day.

We can pose this problem as an optimal control problem as follows: maximize the functional

$$J(u) = \lambda_{min}\left(\int_0^T \frac{\partial y}{\partial x_o}^T \frac{\partial y}{\partial x_o} dt)\right) \tag{2}$$

subject to the constraints

$$\begin{aligned} L &= \int_0^T \sqrt{(\dot{x}_1^2 + \dot{x}_2^2)}\, dt, & (3)\\ \dot{x} &= G(x)u; \quad x(0) = x_o, & (4)\\ y &= h(x). & (5)\end{aligned}$$

The solution of the above optimal control problem might be difficult in general to obtain. In the rest of this paper, we specialize the vehicle kinematic model to be that of an omnidirectional (holonomic) vehicle. We will accordingly disregard the vehicle orientation as a state, and simply assume $G(x)$ in (1) to be the identity matrix. This assumption is equivalent to asking that $G(x)$ be invertible for all x (as it happens in omnidirectional robots), and that a state feedbacl law $u = G^{-1}(x)v$ is applied. This will allow us to use the general results discussed in the next section to the particular type of measurement equations for the problem at hand.

4 Optimal exploratory paths for flat 2D systems

Consider a two-dimensional problem with flat controllability distribution

$$\left\{\begin{array}{l} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ y = h(x_1, x_2) \end{array}\right. \tag{6}$$

with $h(x)$ analytic. Observing that for this system it holds

$$\frac{\partial h}{\partial x_0} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial x_0} \quad \text{with} \quad \frac{\partial x}{\partial x_0} = I$$

and using subscript notation for partial derivatives, we have

$$F = \int_0^T \left[\begin{array}{cc} h_{x_1}^2 & h_{x_1} h_{x_2} \\ h_{x_1} h_{x_2} & h_{x_2}^2 \end{array}\right] dt \in \mathbb{R}^{2\times 2}. \tag{7}$$

The smallest eigenvalue is evaluated as

$$\begin{aligned} \lambda_{min} &= \tfrac{1}{2}\int_0^T h_{x_1}^2 + h_{x_2}^2 dt \\ &-\tfrac{1}{2}\sqrt{\left(\int_0^T h_{x_1}^2 - h_{x_2}^2 dt\right)^2 + 4\left(\int_0^T h_{x_1} h_{x_2} dt\right)^2}. \end{aligned} \tag{8}$$

Observe that, for

$$\frac{h_{x_1}}{h_{x_2}} = C, \tag{9}$$

with C constant, λ_{min} vanishes and, being F semi-positive defined, the curve defined by (9) contains a minimal extremal. Unfortunately, the functional (8) does not enjoy the localization property (i.e., additivity over time, see [10]), so that standard results of the calculus of variations and optimal control cannot be applied directly.

To proceed in the analysis, it is expedient to recall Rayleigh's lemma:

Lemma 1 *Let $A \in \mathbb{R}^{n\times n}$ be symmetric, and λ_M, λ_m denote its largest and smallest eigenvalues. Then λ_M, λ_m are respectively the maximum and the minimum values of the bilinear form*

$$\lambda = x^T A x \quad x \in \mathbb{R}^n$$

on $B(0,1) \in \mathbb{R}^n$.

Based on this, we can trasform the functional (8) as

$$\begin{aligned} \lambda_{min} &= \min_{\|x\|=1} x^T \int_0^T \left[\begin{array}{cc} h_{x_1}^2 & h_{x_1} h_{x_2} \\ h_{x_1} h_{x_2} & h_{x_2}^2 \end{array}\right] dt\ x = \\ &= \min_{\theta\in[0..2\Pi]} \int_0^T (\cos\theta h_{x_1} + \sin\theta h_{x_2})^2 dt. \end{aligned} \tag{10}$$

For a given value θ_0, consider the functional

$$\lambda_{\theta_0} = \int_0^T (\cos\theta_0 h_{x_1} + \sin\theta_0 h_{x_2})^2 dt. \tag{11}$$

This functional (11) does have the localization property and all classic results are applicable. In particular (11) (and hence also (10)) depends on the parametrization chosen for the path (see [10], [11]), or, in other terms, on the velocity at which the path is followed. We will consider henceforth constant velocity, and impose $|\dot{x}| = 1$ along all solutions.

This condition is equivalent (by integration) to

$$\int_0^T \sqrt{\dot{x_1}^2 + \dot{x_2}^2} dt = T. \tag{12}$$

Let γ_M, γ_m be curves in $C^0[0,T]$ which respectively maximize and minimize λ_{min}, and let $\theta_m(\gamma) = \arg\min_\theta \lambda_{min}(F)$ so that

$$\lambda_{min} = \int_0^T (\cos\theta_m(\gamma) h_{x_1} + \sin\theta_m(\gamma) h_{x_2})^2 \, dt. \quad (13)$$

Since the extremum shall verify (12), γ_M, γ_m must extremize the functional

$$\lambda_{min} = \int_0^T (\cos\theta_m(\gamma) h_{x_1} + \sin\theta_m(\gamma) h_{x_2})^2 + \mu\sqrt{\dot{x}_1^2 + \dot{x}_2^2} dt \quad (14)$$

(see [10]). Observe that, since $[cos(\theta), sin(\theta)]^T$ in an eigenvector of F which does not depend on t but only on T, $\theta_m(\gamma)$, and hence also

$$f \stackrel{def}{=} (\cos\theta_m(\gamma) h_{x_1} + \sin\theta_m(\gamma) h_{x_2})^2,$$

do not depend explicitly on t. Then writing Euler's equations for the functional (14),

$$\frac{\partial}{\partial x_1} f - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_1}\sqrt{\dot{x}_1^2 + \dot{x}_2^2} = 0 \quad (15)$$

$$\frac{\partial}{\partial x_2} f - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_2}\sqrt{\dot{x}_1^2 + \dot{x}_2^2} = 0, \quad (16)$$

and observing that

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_1}\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\right) = \frac{d}{dt}\frac{\dot{x}_1}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}} = \frac{\dot{x}_2\,(-\dot{x}_1\,\ddot{x}_2 + \ddot{x}_1\,\dot{x}_2)}{\sqrt{(\dot{x}_1^2 + \dot{x}_2^2)^3}} \quad (17)$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_2}\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\right) = \frac{d}{dt}\frac{\dot{x}_2}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}} = \frac{\dot{x}_1\,(-\dot{x}_2\,\ddot{x}_1 + \ddot{x}_2\,\dot{x}_1)}{\sqrt{(\dot{x}_1^2 + \dot{x}_2^2)^3}} \quad (18)$$

by multiplying (17) by $\dot{x}_1$, (18) by $\dot{x}_2$, and adding up we get

$$\frac{\dot{x}_1\dot{x}_2\,(-\dot{x}_1\,\ddot{x}_2 + \ddot{x}_1\,\dot{x}_2)}{\sqrt{(\dot{x}_1^2 + \dot{x}_2^2)^3}} + \frac{\dot{x}_2\dot{x}_1\,(-\dot{x}_2\,\ddot{x}_1 + \ddot{x}_2\,\dot{x}_1)}{\sqrt{(\dot{x}_1^2 + \dot{x}_2^2)^3}} = 0.$$

Doing the same with (15), (16) we get

$$\begin{aligned} &\dot{x}_1\left(f_{x_1} - \frac{d}{dt}\left(\frac{\partial}{\partial x_1}\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\right)\right) + \\ &+\dot{x}_2\left(f_{x_2} - \frac{d}{dt}\left(\frac{\partial}{\partial x_2}\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\right)\right) \\ &= f_{x_1}\dot{x}_1 - \lambda\dot{x}_1\frac{d}{dt}\left(\frac{\partial}{\partial x_1}\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\right) + \\ &+f_{x_2}\dot{x}_2 - \lambda\dot{x}_2\frac{d}{dt}\left(\frac{\partial}{\partial x_2}\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\right) \\ &= \frac{df}{dt} = 0 \end{aligned}$$

and finally

$$f(x_1, x_2) = C. \quad (19)$$

Recall that C is a constant, hence (19) is a first integral of the functional (14). We have thus proved that extremals of the variational problem lie in the one-parameter (θ) set

$$f(x_1, x_2) = C.$$

Finally, in order to determine C, note that the starting point of the trajectory is known (x_1^0, x_2^0) and hence

$$f(x_1, x_2) = f(x_1, x_2)|_{(x_1^0, x_2^0)}. \quad (20)$$

Replacing f in (20) we have

$$\begin{aligned}&(\cos\theta_m(\gamma) h_{x_1} + \sin\theta_m(\gamma) h_{x_2})^2 = \\ &= (\cos\theta_m(\gamma) h_{x_1}|_{(x_1^0,x_2^0)} + \sin\theta_m(\gamma) h_{x_2}|_{(x_1^0,x_2^0)})^2\end{aligned},$$

which implies either

$$\cos\theta_m(h_{x_1} + h_{x_1}|_{(x_1^0,x_2^0)}) + \sin\theta_m(h_{x_2} + h_{x_2}|_{(x_1^0,x_2^0)}) = 0 \quad (21)$$

or

$$\cos\theta_m(h_{x_1} - h_{x_1}|_{(x_1^0,x_2^0)}) + \sin\theta_m(h_{x_2} - h_{x_2}|_{(x_1^0,x_2^0)}) = 0. \quad (22)$$

It is simple to verify that, if

$$h_{x_1},\ h_{x_1}|_{(x_1^0,x_2^0)},\ h_{x_2},\ h_{x_2}|_{(x_1^0,x_2^0)}$$

are not identically zero, (21) does not contain (x_1^0, x_2^0), and so, extremals must be found in the set described (22). Observe that, setting

$$\tan(\theta_m) = -\frac{h_{x_1}|_{(x_1^0,x_2^0)}}{h_{x_2}|_{(x_1^0,x_2^0)}}$$

in (22), the set (9) is obtained, i.e. for this particular value of θ_m (22) contains the minimal extremal (along which observability is lost). Finally observe that the minimal trajectory does not depend on T.

5 Optimal exploratory paths with triangulation

Consider again the omnidirectional vehicle model $\dot{x} = u$, $x \in \mathbb{R}^2$, and consider its self-localization in an environment containing two markers m_1, m_2 of coordinates $(0, p)$ and $(0, -p)$, respectively. As an output measurement, we consider the angle comprised between the segments $\bar{m_1 x}$ and $x\bar{m}_2$ that can be easily measured by optical triangulation devices. More precisely, the state and output equations are assumed to be

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ h = \frac{x_1^2 + x_2^2 - p^2}{2px_1} \end{cases} \quad (23)$$

where, for simplicity's sake, we take as output function the inverse of the cosine of the angle $\widehat{m_1 x m_2}$, which is defined everywhere except for $x_1 = 0$. Notice that for all ξ with $|\xi| \geq 1$, it holds $h^{-1}(\xi) = C_\xi$ with C_ξ the circle $\{x \in \mathbb{R}^2 | x_1^2 + x_2^2 - p^2 - 2p\xi x_1 = 0\}$ The observability codistribution for this system is given by

$$O = \begin{bmatrix} \frac{1}{2}\frac{x_1^2 - x_2^2 + p^2}{p\,x_1^2} & \frac{x_2}{p\,x_1} \\ \frac{x_2^2 - p^2}{p\,x_1^3} & -\frac{x_2}{p\,x_1^2} \\ -\frac{x_2}{p\,x_1^2} & \frac{1}{x_1\,p} \\ \vdots & \vdots \end{bmatrix} \tag{24}$$

and it has full rank on $\mathbb{R}^2 \setminus \{x = 0\}$ for $p \neq 0$. Hence the system (23) is locally observable.

Writing expression (22) for system (23), we obtain:

$$\cos\theta_m\left(\frac{1}{2}\frac{x_1^2 - x_2^2 + p^2}{p\,x_1^2} - \frac{1}{2}\frac{(x_1^0)^2 - (x_2^0)^2 + p^2}{p\,(x_1^0)^2}\right) + \sin\theta_m\left(\frac{x_2}{p\,x_1} - \frac{x_2^0}{p\,(x_1^0)}\right) = 0.$$

Without loss of generality, let us set $p = 1$ and introduce the notation

$$(h_{x_1}|_{(x_1^0,x_2^0)}, h_{x_2}|_{(x_1^0,x_2^0)}) = (q_1, q_2).$$

Relation (22) becomes:

$$\begin{array}{l}(cos(\theta_m) - 2cos(\theta_m)q_1 - 2sin(\theta_m)q_2)x_1^2 \\ \quad +2sin(\theta_m)x_1x_2 - cos(\theta_m)x_2^2 + cos(\theta_m) = 0\end{array} \tag{25}$$

Dividing by $sin(\theta_m) \neq 0$, we get

$$(-2\alpha q_1 + \alpha - 2q_2)x_1^2 + 2x_1x_2 - \alpha x_2^2 + \alpha, \tag{26}$$

where $\alpha = \text{cotan}\,(\theta_m)$. Relation (9) becomes

$$-x_1^2 + x_2^2 - 1 + 2q_1x_1^2 - 2x_1x_2 + 2q_2x_1^2 = 0$$

which is a special case of (26). Finding maximal extremals is more complex. By the conics classification theorem (see [9]), writing (26) as

$$\begin{bmatrix} x_1 & x_2 & 1 \end{bmatrix} \begin{bmatrix} 2\alpha q_1 - \alpha + 2q_2 & -1 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & -\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = 0, \tag{27}$$

and setting

$$\alpha_1 = \frac{1}{2}\frac{-2\,q_2 + 2\sqrt{q_2{}^2 - 2\,q_1 + 1}}{2\,q_1 - 1},$$

$$\alpha_2 = \frac{1}{2}\frac{-2\,q_2 - 2\sqrt{q_2{}^2 - 2\,q_1 + 1}}{2\,q_1 - 1},$$

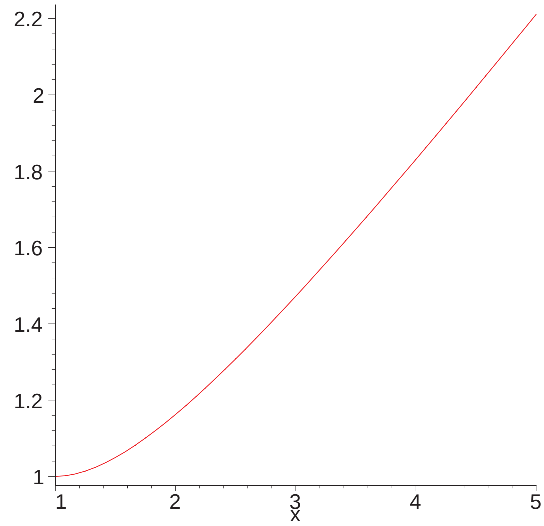

Figure 3: The minimal extremal starting from $(1, 1)$ is a hyperbola. Moving along this path minimizes observability (actually, it makes the problem unobservable).

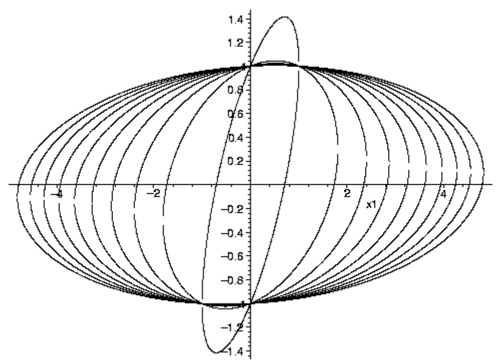

Figure 4: Conics starting from $(1, 1)$ and for $\alpha > 1/2$

we have an ellipse if $\alpha > \alpha_1$ or $\alpha < \alpha_2$; a hyperbola for $\alpha_2 < \alpha < \alpha_1$, and a pair of straight lines for $\alpha = \alpha_1,\ \alpha_2$. Because of the choice in (12), we need an arc length (natural) parametrization of (27).

Let us start by observing that, by the change of coordinates

$$R = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$\tan(\beta) = \alpha\,q_1 + q_2 + \sqrt{\alpha^2\,q_1{}^2 + 2\,\alpha\,q_1\,q_2 + q_2{}^2 + 1}$$

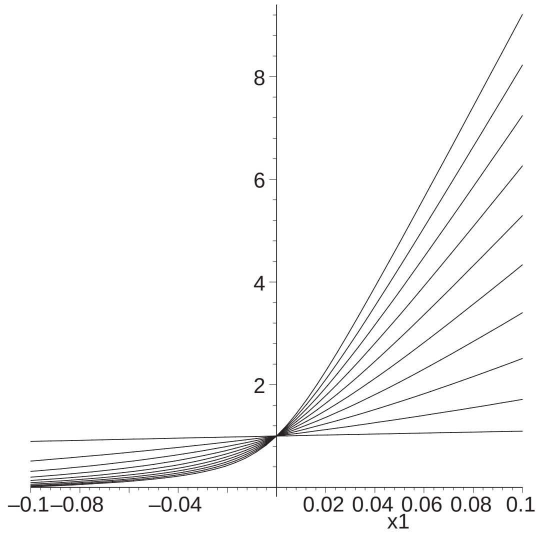

Figure 5: Conics starting from $(1,1)$ and for $\alpha < 1/2$

conics are rewritten in principal axes as

$$= \begin{bmatrix} x_1' & x_2' & 1 \end{bmatrix} \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & -\alpha \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = 0 \tag{28}$$

with

$$A_1 = -\alpha\, q_1 + \alpha - q_2 + \sqrt{\alpha^2\, q_1{}^2 + 2\,\alpha\, q_1\, q_2 + q_2{}^2 + 1}$$

$$A_2 = -\alpha\, q_1 + \alpha - q_2 - \sqrt{\alpha^2\, q_1{}^2 + 2\,\alpha\, q_1\, q_2 + q_2{}^2 + 1}$$

Given an arc length parametrization for (28), by applying R^{-1} an arc length parametrization for (27) is obtained (arclength parametrizations are isometry-invariant). Recall the definitions of the elliptic functions of $z \in \mathbb{R}$ (parametrized by $k \in \mathbb{R}$):

$$EllipticF(z,k) = \int_0^z \frac{1}{\sqrt{1-k^2t^2}\sqrt{1-t^2}}\, dt \tag{29}$$

(elliptic integrals of the first kind)

$$EllipticE(z,k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt \tag{30}$$

(elliptic integrals of the second kind). A natural parametrization for the ellipse (28) is given by

$$\begin{cases} x' = \sqrt{\frac{\alpha}{A_1}} cos(El^{-1}(s)) \\ y' = \sqrt{\frac{\alpha}{A_2}}\, sin(El^{-1}(s)) \end{cases} \tag{31}$$

where

$$s = El(t) = -\sqrt{\frac{\alpha}{A_1}}\ (EllipticE(cos\ t, \sqrt{\frac{\alpha}{A_2} - 1}))$$
$$+\sqrt{\frac{\alpha}{A_1}}(EllipticE(1, \sqrt{\frac{\alpha}{A_2} - 1}))-$$

(for further details see [11]). By replacing

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cos(\beta) & sin(\beta) \\ -sin(\beta) & cos(\beta) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

in λ_m, and deriving with respect to α (notice that only elliptic functions with their inverse appear in $\partial\lambda_m/\partial\alpha$), we obtain extremals corresponding to values of α that solve $\partial\lambda_m/\partial\alpha = 0$. This is a polynomial equation in $\tan(\theta_m)$, which is divisible by

$$tan(\theta_m) + \frac{h_{x_1}|_{(x_1^0,x_2^0)}}{h_{x_2}|_{(x_1^0,x_2^0)}} = 0$$

(indeed we know this to be a minimal extremal). Maximal extremals are then obtained by solving (numerically) the resulting simplified polynomial equation.

The hyperbolic case is analogous, by substituting relation (31) with

$$\begin{cases} x' = (A_2/A_1)\sqrt{(Il^{-1}(s))^2 + \alpha^2/A_2^2} \\ y' = Il^{-1}(s) \end{cases} \tag{32}$$

where

$$s = Il(t) = \frac{\text{EllipticE}(\frac{A_2 t}{\alpha}, \sqrt{\frac{(\alpha/A_2)^2 + (\alpha/A_1)^2}{(\alpha/A_2)^2}})}{(\alpha/A_2)} -$$
$$\frac{\text{EllipticE}(\frac{A_2}{\alpha}, \sqrt{\frac{(\alpha/A_2)^2 + (\alpha/A_1)^2}{(\alpha/A_2)^2}})}{(\alpha/A_2)}$$

for a hyperbola in the form (28).

Some examples of optimal paths of different length are reported in fig.5. Notice that the optimal paths differ for different lengths.

6 Conclusion

In this paper we have considered a basic problem behind planning exploratory motions in an unknown environment, that is, the geometry of paths that achieve maximum information for a given length traversed. Results show that optimal paths are arcs of conics: depending on the starting point, they can be either hyperbola, ellipses, or straight lines. For a given starting point, the optimal conic depends only on the assigned length of the path to be traversed.

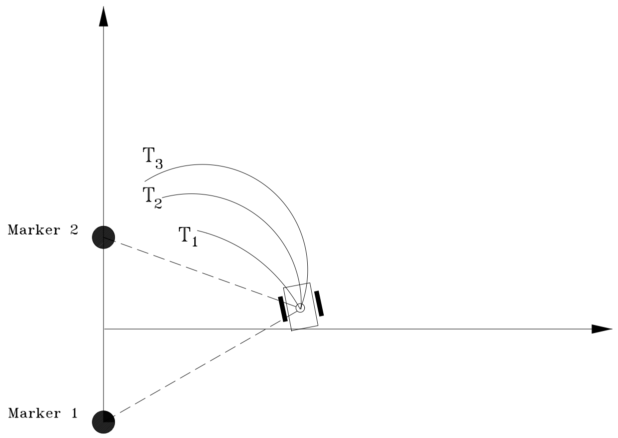

Figure 6: Optimal trajectories for three different path lengthsT_1, T_2, T_3

Although our results only apply to a simple model of omnidirectional vehicles and triangulation measurements, we believe that it provides some insight in the practically important problem of optimally planning exploratory motions of given length for more general systems, which problems will be the objective of further studies.

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