A Local Separation Property of Locally Observable Analytic Nonlinear Systems

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Abstract

This paper presents a novel nonlinear observer, which exhibits a local separation property. In fact, if there exists a stabilizing static state feedback, the designed observer permits to achieve local practical stability of the closed-loop system, if the real state has been substituted with the current estimated one. The observer requires only that the nonlinear system must be locally observable for the considered real analytic input function. A strategy based on the use of redundant observables, i.e. estimated higher order output derivatives, permits to deal with bad inputs.

Keywords: Nonlinear control systems, Nonlinear observability, Observer design, Local separation property, Bad inputs

1 Introduction

1.1 Main Results

In this paper, we consider general nonlinear systems of the form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{u}) \\ \mathbf{y}(\mathbf{x}) &= \mathbf{h}(\mathbf{x}) \quad , \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbf{X}$, an open subset of $\Re^n$ containing the origin $\mathbf{x} = 0$, is the state vector, $\mathbf{u} \in \Re^m$ is the control input vector, $\mathbf{y} \in \Re^p$ is the output vector. The vector field $\mathbf{f}(\mathbf{x},\mathbf{u})$ and the output map $\mathbf{h}(\mathbf{x})$ are assumed real analytic in the following, and $\mathbf{f}(0,0) = 0$.

The main contribution of this paper is twofold:

- a novel local nonlinear observer is presented, which ensures practical stability of the trivial equilibrium in the observation error dynamics;
- a local separation property is achieved for the considered class of nonlinear analytic system, i.e. locally observable systems in the sense of [27, 30].

The main idea in the observer design is the use of higher-order output time derivatives (henceforth called "observables") taken from the observability space associated to (1) in the assumption that the rank condition on the observability matrix is satisfied for the considered real analytic input function. Connections with strong observability under piecewise constant inputs are also highlight. Then, the observables of order higher than one are estimated in the observer design by using high-pass filters. Practical stability is guaranteed since the introduced persistent perturbations can be made arbitrarily small.

Moreover, a local separation property for the considered class of nonlinear system is proven. In particular if there exists a stabilizing state feedback, the equilibrium with the estimated state feedback remains locally practically stable. We also consider the problem of *bad inputs*, which has not been satisfactorily treated in the literature yet. We associate a so-called singularity manifold to the nonlinear system (1), the use of a redundant number of observables ensures a well-conditioned EOJ generalized inversion during the state regulation. Finally an example is reported. The considered system is drift-less, and not uniformly locally observable. We stress that the techniques proposed in [22, 1] can not be applied to the presented example. Our approach ensures local practical output stabilization of the trivial equilibrium. Even if the proposed analysis is local, the simulation results show that the region of attraction has a noticeable extension.

The paper is organized as follows. In Sect. 2, local observability of analytic nonlinear systems is briefly reminded. Sect. 3 presents a local separation property, which derives naturally from the application of the proposed nonlinear observer. In Sect. 4, a redundant observer design is derived to deal with observability singularities. In Sect. 5 the proposed framework has been applied to a simple, but meaningful, example. In Sect. 6 the major contribution of the paper is summarized and future investigations are outlined.

1.2 Related Work

A first approach to design an observer is to transform the original nonlinear system into another one for which the design is known. Transformations, which have been proposed in the literature, are the system immersion [8] which permits to obtain a bilinear system if the observation space is finite dimensional, and the linearization by means output injection [16, 17, 19] assuming that particular differential–geometric conditions on the system vector fields are verified. Rank conditions under which the dynamics of the observation error is linear, i.e. the original system can be transformed into the observer canonical form, are also investigated in [27]. Results on bilinear observers are presented in [5, 10]. Extension of the Luenberger filter in a nonlinear setting, by using the time derivatives of the input, has been proposed in [30].

Early results on the observer design of bilinear systems without bad inputs are reported in [29]. Gauthier et al. [9, 3] generalized the results in the case of input-affine nonlinear systems without bad inputs and applied the approach to biological reactors. The first step is to write the input affine nonlinear system in a so-called normal observation form. However, this form requires that the trivial input is an universal input [2] for the system, and also that a diffeomorphism can be constructed using the Lie derivatives of the output along the drift nonlinear term. Results on the normal observation form have been provided also by Tsinias [25, 26]. In [6] the authors consider single input - single output input-affine nonlinear systems, and in the case of relative degree equal to the dimension of the state space n, full-rankness of the observability matrix, and global Hödel conditions on appropriate functions, it is shown the global asymptotic convergence of the estimated state, while in case of relative degree less than n, stronger conditions on the admissible inputs have to be assumed.

High-gain techniques have been applied in the field of nonlinear observers. Early results are due to Tornambè [24], which proposed an approach based on high-gain approximate cancellation of the nonlinearity. A high gain observer which estimates the output derivatives combined with a globally bounded state feedback control law permits to obtain semiglobal stabilization by output feedback [7, 14, 20, 22, 23] in case of uniformly observable input-affine nonlinear systems [22]. In the recent reference [1] the authors employ a separation principle of a certain class of nonlinear systems, showing that with the estimated state feedback it is possible the performance recovery of the real state feedback, i.e. the asymptotic stability of the equilibrium, the region of attraction, and trajectories. Even if the results obtained concern stability analysis in the large, the class of considered systems is quite restricted. As we will show in the following, the approach presented in [1] cannot be applied to the presented example.

2 The Class of Systems

Let us associate to the nonlinear system (1) the following extended output map $\Phi : \Re^n \times \Re^{l_p\, m} \to \Re^{(l_p+1)\, p}$, defined as:

$$\Phi(\mathbf{x},\mathbf{v}) = \begin{pmatrix} \phi_0(\mathbf{x}) \\ \phi_1(\mathbf{x},\mathbf{v}_0) \\ \dots \\ \phi_{l_p}(\mathbf{x},\mathbf{v}_0,\dots,\mathbf{v}_{l_p-1}) \end{pmatrix} \tag{2}$$

where $\mathbf{v} = [\mathbf{v}_0^T,\dots,\mathbf{v}_{l_p-1}^T]^T = [\mathbf{u}^T,\dots,\mathbf{u}^(l_p-1)^T]^T \in \Re^{l_p\, m}$ is the extended input vector (i.e. the input vector and its time derivatives up to order l_p-1), and l_p is an integer such that $(l_p+1)\, p \geq n$. The observable of order $i=0,\dots,l_p$ is $\phi_i(\mathbf{x},\mathbf{v}_0,\dots,\mathbf{v}_{i-1})$, the $i-th$ output time derivative. These functions are defined recursively as:

$$\begin{aligned} \phi_0(\mathbf{x}) &= \mathbf{h}(\mathbf{x}) \\ \phi_1(\mathbf{x},\mathbf{v}_0) &= \frac{\partial \phi_0(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{v}_0) \\ \dots & \\ \phi_{l_p}(\mathbf{x},\mathbf{v}_0,\dots,\mathbf{v}_{l_p-1}) &= \frac{\partial \phi_{l_p-1}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{v}_0) \\ &+ \sum_{j=0}^{l_p-2}\frac{\partial \phi_{l_p-1}}{\partial \mathbf{v}_j}\mathbf{v}_{j+1} \quad . \end{aligned} \tag{3}$$

Assumption 1 *Fixed an integer l_p such that $(l_p+1)\, p \geq n$, the map $\Phi(\mathbf{x},\mathbf{v})$ satisfies the rank condition:*

$$rank\left(\frac{\partial \Phi(\mathbf{x},\mathbf{v})}{\partial \mathbf{x}}\right) = n \quad , \tag{4}$$

$\forall(\mathbf{x},\mathbf{v})$ *in an open neighborhood* $\mathbf{X}^0 \times \mathbf{V}^0$ *of* $\mathbf{X}\times\Re^{l_p\, m}$.

We define $\mathbf{J}(\mathbf{x},\mathbf{v}) = \left(\frac{\partial \Phi(\mathbf{x},\mathbf{v})}{\partial \mathbf{x}}\right)$ and call this matrix the Extended Output Jacobian (EOJ) associated to the nonlinear system (1).

We now give the following motivation of the introduced assumption, based on the Implicit Function Theorem. Let us introduce the map $\mathbf{F} : \Re^{[(l_p+1)\, l_p\, p\, m]} \times \Re^n \to \Re^{(l_p+1)\, p}$, defined as:

$$\mathbf{F}(\overline{\mathbf{z}},\overline{\mathbf{x}}) = \begin{pmatrix} \overline{\mathbf{z}}_0 - \phi_0(\overline{\mathbf{x}}) \\ \overline{\mathbf{z}}_1 - \phi_1(\overline{\mathbf{x}},\overline{\mathbf{z}}_{l_p+1}) \\ \dots \\ \overline{\mathbf{z}}_{l_p} - \phi_{l_p}(\overline{\mathbf{x}},\overline{\mathbf{z}}_{l_p+1},\dots,\overline{\mathbf{z}}_{2\, l_p}) \end{pmatrix} \quad , \tag{5}$$

where $\overline{\mathbf{z}}_k \in \Re^p$, $k=0,\dots,l_p$, $\overline{\mathbf{z}}_k \in \Re^m$, $k = l_p+1,\dots,2\, l_p$, and $\overline{\mathbf{x}} \in \Re^n$. Consider the extended nonlinear system associated to (1):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{v}_0) \tag{6}$$

$$\begin{aligned}\dot{\mathbf{v}}_0 &= \mathbf{v}_1\\ &\dots\\ \dot{\mathbf{v}}_{l_p-1} &= \nu\\ \xi &= \Phi(\mathbf{x},\mathbf{v}) \quad ,\end{aligned}$$

where $(\mathbf{x},\mathbf{v}) \in \mathbf{X} \times \Re^{l_p m}$ is the extended state vector, $\nu \in \Re^m$ is the control input vector, and $\xi \in \Re^{(l_p+1)p}$ is the extended output vector. At any time instant $t \geq t^0 \geq 0$ the flow of the above nonlinear system satisfies the equation:

$$\mathbf{F}(\mathbf{z},\mathbf{x}) = 0 \quad , \tag{7}$$

where $\mathbf{z} = [\xi^T \ \mathbf{v}^T]^T \in \Re^{l_p (l_p+1) p m}$, $\xi = [\xi_0^T, \dots, \xi_{l_p}^T] \in \Re^{(l_p+1)p}$, and $\mathbf{x} \in \mathbf{X}$. If the assumption 1 is satisfied, given a point $(\mathbf{z}^0, \mathbf{x}^0)$, $\mathbf{z}^0 = [\xi^{0\,T} \ \mathbf{v}^{0\,T}]^T$, such that $(\mathbf{x}^0, \mathbf{v}^0) \in \mathbf{X}^0 \times \mathbf{V}^0$, there exist n observables taken from the $(l_p+1)\,p$ ones in $\Phi(\mathbf{x},\mathbf{v})$ which are linear independent in $\mathbf{X}^0 \times \mathbf{V}^0$. By the Implicit Function Theorem, there exist two open neighborhoods, namely $\mathbf{A}^0 = \mathbf{I}^0 \times \mathbf{V}^0$ of $\mathbf{z}^0$, $\mathbf{X}^0$ of $\mathbf{x}^0$, being $\mathbf{I}^0$ an open neighborhood of ξ^0, an unique map $\mathbf{g} : \mathbf{A}^0 \to \mathbf{X}^0$, with $\mathbf{g}(\mathbf{z}) = \mathbf{x}$, such that $\mathbf{F}(\mathbf{z}, \mathbf{g}(\mathbf{z})) = 0$.

Remark 1 *In the case of an uniformly observable SISO (Single Input Single Output) nonlinear system [22, 18], given the map* $[y \ \dot{y} \ \dots y^{(n-1)}]^T = \Phi(\mathbf{x},\mathbf{v}) \in \Re^n$, *being* $\mathbf{x} \in \Re^n$ *the state vector, and* $\mathbf{v} = [u \ \dot{u} \ \dots u^{(n-2)}]^T \in \Re^{n-1}$ *the extended input vector, the Assumption 1 is satisfied for each* $(\mathbf{x},\mathbf{v}) \in \Re^n \times \Re^{n-1}$. *Thus, uniform observability is a sufficient condition for our observer design.*

Assume that the input function $\mathbf{u}(t)$, $t \geq t_0 \geq 0$ of the considered nonlinear system (1) is real analytic. The nonlinear system can be viewed as a time-varying analytic system without input:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{u}(t)) = \overline{\mathbf{f}}(\mathbf{x},t) &(8)\\ \mathbf{y}(\mathbf{x}) &= \mathbf{h}(\mathbf{x}) \quad . &(9)\end{aligned}$$

We remind the following definition [13, 30, 27].

Definition 1 *The system (8) is locally observable at* $\mathbf{x}^0$ *in the interval* $[t_0, \ T]$, $\mathbf{x}^0 \in \mathbf{X}$, $T > t_0 \geq 0$, *if given the output function* $\mathbf{y}(t)$, $t \in [t_0, \ T]$, *then* $\mathbf{x}^0$ *can be uniquely distinguished in a small neighborhood.*

The class of nonlinear systems which are locally observable at $\mathbf{x}^0$ in the interval $[t_0, \ T]$ is determined by the following proposition [27, 30].

Proposition 1 *The system (8) is locally observable at* $\mathbf{x}^0$ *in the interval* $[t_0, \ T]$ *if and only if there exist a neighborhood* $\overline{\mathbf{X}}^0$ *of* $\mathbf{x}^0$, *and an p-tuple of integers* $(k_1, \dots, k_p)$, *called the observability indices, such that:*

- $k_1 \geq k_2 \geq k_p > 0$ *and* $\sum_{i=1}^p k_i = n$;
- *defined the differential operator:* $\mathcal{N}^0 \mathbf{w} = \mathbf{w}$, *and*

$$\mathcal{N}\mathbf{w} = \overline{\mathbf{f}}^T \frac{\partial \mathbf{w}^T}{\partial \mathbf{x}} + \mathbf{w}\frac{\partial \overline{\mathbf{f}}}{\partial \mathbf{x}} + \frac{\partial \mathbf{w}}{\partial t}$$

where $\mathbf{w}(\mathbf{x},t) = (w_1(\mathbf{x},t), \dots, w_p(\mathbf{x},t))$, $w_i(\mathbf{x},t)$ *are real analytic time-varying functions, the observability matrix* $\mathbf{Q}(\mathbf{x},t) \in \Re^{n \times n}$ *defined as:*

$$\mathbf{Q}(\mathbf{x},t) = \begin{pmatrix} dh_1(\mathbf{x})\\ \mathcal{N} dh_1(\mathbf{x})\\ \dots\\ \mathcal{N}^{(k_1-1)} dh_1(\mathbf{x})\\ \dots\\ dh_p(\mathbf{x})\\ \dots\\ \mathcal{N}^{(k_p-1)} dh_p(\mathbf{x}) \end{pmatrix} \quad , \tag{10}$$

is nonsingular $\forall \mathbf{x} \in \overline{\mathbf{X}}^0$, *and* $t \in [t_0, \ T]$, *being* $dh_i = \frac{\partial h_i}{\partial \mathbf{x}}$, $i = 1, \dots, p$ *the exact differentials associated to the output.*

The proof of the above proposition uses results of linear time-varying systems [21], and of perturbed differential equations [12], see also [13, 27].

We are now in the position to determine the class of nonlinear analytic systems which satisfies Assumption 1.

Proposition 2 *Assume that the input function* $\mathbf{u}(t)$, $t \geq t_0 \geq 0$ *of the analytic nonlinear system (1) is real analytic. Assumption 1 holds if and only if the system (8) is locally observable at every* $\mathbf{x}^0 \in \mathbf{X}^0$ *in the interval* $[t_0, \ T]$, *for some* $T > t_0 \geq 0$.

Proof: If Assumption 1 holds, by using the Implicit Function Theorem, for every real analytic input function, given the output $\mathbf{y}(t)$, $t \geq t_0 \geq 0$, there exists an unique function $\mathbf{x}(t)$, $t \geq t_0 \geq 0$ which undergoes the equation $\mathbf{F}(\mathbf{z},\mathbf{x}) = 0$. This is in fact the unique implicit function $\mathbf{g}(\mathbf{z}) = \mathbf{x}$, such that $\mathbf{F}(\mathbf{z}, \mathbf{g}(\mathbf{z})) = 0$. Hence, the system (8) is locally observable at every $\mathbf{x}^0 \in \mathbf{X}^0$ in the interval $[t_0, \ T]$, for some interval $T - t_0 > 0$ sufficiently small. Notice that since the rows of the observability matrix $\mathbf{Q}(\mathbf{x},t)$ are differentials which appear as rows in the EOJ matrix $\mathbf{J}(\mathbf{x},\mathbf{v})$, after a possible reordering of the indices of the output variables, there exist the observability indices which satisfy the conditions of Prop. 1 in $[t_0, \ T]$.

Vice versa, if the system (8) is locally observable at every $\mathbf{x}^0 \in \mathbf{X}^0$ in the interval $[t_0, \ T]$, for some interval $T - t_0 > 0$ sufficiently small, then by choosing $l_p =$

$k_1 - 1$, where k_1 is the higher observability index, then Assumption 1 holds by construction. ■

The concept of local observability at $\mathbf{x}^0$ in the time interval $[t_0,\ T]$, $T > t_0$ with the assumption of real analytic input can be related to the observability of nonlinear systems under piecewise constant input. We remind that [28] the nonlinear system (1) is strongly observable at $\mathbf{x}^0$, if the autonomous system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}^{\overline{\mathbf{u}}}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, \overline{\mathbf{u}}) \quad (11) \\ \mathbf{y}(\mathbf{x}) &= \mathbf{h}(\mathbf{x}) \quad . \quad (12) \end{aligned}$$

with $\overline{\mathbf{u}}$ constant, is locally weakly observable [11], for all $\overline{\mathbf{u}}$ of interest. In the analytic case, if the nonlinear system is weakly controllable, then the system is local weakly observable at $\mathbf{x}^0 \in \mathbf{X}^0$ if and only if the observability rank condition is satisfied [11]. If this is the case, strong observability implies that $\mathrm{d}im\,(d\mathcal{O})(\mathbf{x}^0) = n$, $\forall \overline{\mathbf{u}}$ of interest. If a system is strongly observable at $\mathbf{x}^0 \in \mathbf{X}^0$, then every constant input function distinguishes between nearby states, which imply that every C^∞ function (and in particular analytic) also distinguishes (see [28]). Hence if the nonlinear system(1) is strongly observable at every $\mathbf{x}^0 \in \mathbf{X}^0$ then there exists $T > t_0$ such that it is locally observable at $\mathbf{x}^0$ in the interval $[t_0,\ T]$ under real analytic input. We can state the following.

Proposition 3 *Assume that the system (1) is weakly controllable and analytic. If the system is strongly observable at every $\mathbf{x}^0 \in \mathbf{X}^0$ (under piecewise constant inputs), i.e. $dim\,(d\mathcal{O})(\mathbf{x}^0) = n$, $\forall \overline{\mathbf{u}}$ of interest, then the system (1), under real analytic input functions, satisfies Assumption 1.*

Proof: This is a straightforward consequence of the above discussion and Prop. 2. ■

3 A Local Separation Property

Assume that there exists a static state feedback which ensures the local asymptotic stability of the trivial equilibrium.

Assumption 2 *Consider the nonlinear system (1), there exist two functions $\alpha : \mathbf{X} \rightarrow \Re^m$, and $V : \mathbf{X} \rightarrow \Re$ both, at least, of class C^1, such that $V(\mathbf{x})$ is positive definite, and $\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \alpha(\mathbf{x})) < 0$ in an open neighborhood of the origin $\mathbf{x} = 0$.*

We remind the following result, for the proof, see for example [4, 18, 15].

Proposition 4 *Consider the extended nonlinear system in Eq. (6), if Assumption 2 holds, then there exists a function $\overline{\alpha} : \mathbf{X} \times \Re^{l_p m} \rightarrow \Re^m$ of class, at least, C^1, such that the equilibrium $(\mathbf{x}, \mathbf{v}) = (0, 0)$ of the closed-loop system:*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{v}_0) \quad (13) \\ \dot{\mathbf{v}}_0 &= \mathbf{v}_1 \\ &\cdots \\ \dot{\mathbf{v}}_{l_p-1} &= \overline{\alpha}(\mathbf{x}, \mathbf{v}) \quad , \end{aligned}$$

is asymptotically stable.

Denote with s the Laplace variable, $\mathbf{Y}(s)$ the output Laplace transform. $\hat{\xi} = [\mathbf{y}^T,\ \psi_1^T, \ldots, \psi_{l_p}^T]^T$ is the estimated extended output vector, being $\psi_i \in \Re^p$, $i = 1, \ldots, l_p$ the estimation of the $i-th$ output derivative, and T is a small positive constant. Indicate with $\mathbf{Q}$ a positive definite matrix, and, in virtue of Assumption 1, with $\mathbf{J}^+ = (\mathbf{J}^T\,\mathbf{J})^{-1}\,\mathbf{J}^T$ the left pseudo-inverse of the EOJ $\mathbf{J}^+(\hat{\mathbf{x}}, \mathbf{v})$. The main result of this paper is the following.

Proposition 5 *If Assumptions 1, and 2 hold, and T is chosen sufficiently small, the equilibrium $(\mathbf{x}, \hat{\mathbf{x}} - \mathbf{x}) = (0, 0)$ of the closed loop system:*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{v}_0) \quad (14) \\ \dot{\mathbf{v}}_0 &= \mathbf{v}_1 \\ &\cdots \\ \dot{\mathbf{v}}_{l_p-1} &= \overline{\alpha}(\hat{\mathbf{x}}, \mathbf{v}) \\ \mathbf{y}(\mathbf{x}) &= \mathbf{h}(\mathbf{x}) \\ \dot{\hat{\mathbf{x}}} &= \mathbf{f}(\hat{\mathbf{x}}, \mathbf{v}_0) + \mathbf{P}(\hat{\mathbf{x}}, \mathbf{v})\,(\hat{\xi} - \Phi(\hat{\mathbf{x}}, \mathbf{v})) \\ \mathbf{P}(\hat{\mathbf{x}}, \mathbf{v}) &= (\mathbf{Q} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\hat{\mathbf{x}}, \mathbf{v}_0)\right))\,\mathbf{J}^+(\hat{\mathbf{x}}, \mathbf{v}) \\ \psi_i(s) &= \frac{s^i}{(1 + T\,s)^i}\,\mathbf{Y}(s),\ \psi_i(0) = 0,\ i = 1, \ldots, l_p \end{aligned}$$

is locally practically stable, i.e. $\forall \epsilon > 0$ there exist $\delta_1 > 0$, $K > 0$, and T which depends on K, such that if $\|[\mathbf{x}^T(0), \hat{\mathbf{x}}^T(0) - \mathbf{x}^T(0)]^T\| < \delta_1$, then $\|\epsilon^*(t)\| < K$, $t > 0$, and the solutions of (14) satisfy the condition $\|[\mathbf{x}^T(t, 0, \epsilon^*),\ \hat{x}^T(t, 0, \epsilon^*) - x^T(t, 0, \epsilon^*)]^T\| < \epsilon$, $t > 0$.

We begin with the following Lemmas.

Lemma 1 *Consider the nonlinear system $\mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t)$, where $\mathbf{x} \in \Re^n$, $\mathbf{u}(t) \in \Re^m$, and $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$ are smooth vector fields. Assume that the origin of $\mathbf{x} = \mathbf{f}(\mathbf{x})$ is a locally asymptotically stable equilibrium. Then $\forall \epsilon > 0$, there exist $\delta_1 > 0$ and $K > 0$ such that if $\|\mathbf{x}(0)\| < \delta_1$ and $\|\mathbf{u}(t)\| < K$, $t \geq t^0 \geq 0$, the solution $\mathbf{x}(t, t^0, \mathbf{u})$ of $\mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t)$ satisfies the condition: $\|\mathbf{x}(t, t^0, \mathbf{u})\| < \epsilon$, $t \geq t^0 \geq 0$.*

For the proof see, for example, [18].

Consider the observer:

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{f}(\hat{\mathbf{x}},\mathbf{v}_0)+\mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\,(\hat{\xi}-\Phi(\hat{\mathbf{x}},\mathbf{v})) \qquad (15)\\ \mathbf{P}(\hat{\mathbf{x}},\mathbf{v}) &= (\mathbf{Q}+\left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\hat{\mathbf{x}},\mathbf{v}_0)\right))\,\mathbf{J}^{+}(\hat{\mathbf{x}},\mathbf{v})\\ \psi_i(s) &= \frac{s^i}{(1+T\,s)^i}\,\mathbf{Y}(s).\ \ \psi_i(0)=0,\quad i=1,\ldots,l_p\end{aligned}$$

The above equation can be rewritten as:

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}},\mathbf{u})+\mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\,(\xi-\Phi(\hat{\mathbf{x}},\mathbf{v}))-\mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\epsilon^* \quad ,$$

where $\epsilon^*=[0^T,\ \epsilon_1^T,\ldots,\epsilon_{l_p}^T]^T\in\Re^{(l_p+1)\,p}$, $\epsilon_i=\mathbf{y}^{(i)}-\psi_i,\ i=1,\ldots,l_p$ is the introduced persistent perturbation due to the estimated observables.

Lemma 2 *Consider the nonlinear system (1), if Assumption 1 holds, and T is chosen sufficiently small, the equilibrium $\mathbf{e}=\hat{\mathbf{x}}-\mathbf{x}=0$ of the observation error dynamics deriving from the observer (15) is locally practically stable, i.e. $\forall\epsilon_{\mathbf{e}}>0$ there exist $\delta_1>0$, $K>0$, and T which depends on K, such that if $\|\mathbf{e}(0)\|<\delta_1$, then $\|\epsilon^*(t)\|<K,\ t>0$, and the observation error satisfies the condition $\|\mathbf{e}(t,0,\epsilon^*)\|<\epsilon_{\mathbf{e}},\ t>0$.*

Proof: By using Eq. (15), the dynamics of the observation error $\mathbf{e}=\hat{\mathbf{x}}-\mathbf{x}$ results:

$$\begin{aligned}\dot{\mathbf{e}} &= \left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\hat{\mathbf{x}},\mathbf{v}_0)\right)-\mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\,\left(\frac{\partial\Phi}{\partial\mathbf{x}}\,(\hat{\mathbf{x}},\mathbf{v})\right) \qquad (16)\\ &+ (\mathbf{p_f}(\mathbf{v}_0,\mathbf{e})+\mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\,\mathbf{p}_\Phi(\mathbf{v},\mathbf{e}))\\ &= -\mathbf{Q}\mathbf{e}-\mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\epsilon^*\\ &+ \mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\,\mathbf{p}_\Phi(\mathbf{v},\mathbf{e})-\mathbf{p_f}(\mathbf{v}_0,\mathbf{e})\end{aligned}$$

where ϵ^* is the introduced perturbation due to the observables estimation, and the functions $\mathbf{p_f}(\mathbf{v}_0,\mathbf{e})$ and $\mathbf{p}_\Phi(\mathbf{v},\mathbf{e})$ vanish at $\mathbf{e}=0$ with their first order partial derivatives, i.e. satisfy the conditions:

$$\lim_{\mathbf{e}\to 0}\frac{\mathbf{p_f}(\mathbf{v}_0,\mathbf{e})}{\|\mathbf{e}\|}=0,\quad \lim_{\mathbf{e}\to 0}\frac{\mathbf{p}_\Phi(\mathbf{v},\mathbf{e})}{\|\mathbf{e}\|}=0 \quad . \qquad (17)$$

We now prove by induction on the order of the output derivatives that the error ϵ^* can be reduced to an arbitrarily small perturbation if the constant T is sufficiently small. In fact, the Laplace transform of ϵ_1 results: $\epsilon_1(s)=\dot{\mathbf{Y}}(s)-\frac{s}{1+T\,s}\,\mathbf{Y}(s)=T\frac{s}{1+T\,s}\,\dot{\mathbf{Y}}(s)-\frac{\mathbf{y}(0^+)}{1+T\,s}$, where $\mathbf{Y}(s)$ denotes the output Laplace transform, and (with an abuse of notation) $\dot{\mathbf{Y}}(s)=s\,\mathbf{Y}(s)-\mathbf{y}(0^+)$ indicates the output derivative Laplace transform. $\forall t\geq 0$, $\epsilon_1(t)=T\,\dot{\mathbf{y}}_f(t)-\frac{1}{T}\,\mathbf{y}(0^+)\,\exp(\frac{-t}{T})$, where $\dot{\mathbf{y}}_f(t)$ is the output derivative filtered by $\frac{s}{1+T\,s}$. Since the high-pass filter $\frac{s}{1+Ts}$ is Bounded Input Bounded Output (BIBO), there exists $M_{\dot{\mathbf{y}}}>0$ such that $\|\dot{\mathbf{y}}_f(t)\|<M_{\dot{\mathbf{y}}},\ \forall t\geq 0$. Let us consider the term $\frac{1}{T}\,\mathbf{y}(0^+)\,\exp(\frac{-t}{T})$ in the output derivative error $\epsilon_1(t)$, it holds: $\lim_{T\to 0}\frac{1}{T}\,\|\mathbf{y}(0^+)\|\,\exp(\frac{-t}{T})=0,\ \forall t>0$, which means that $\forall\epsilon^*>0$, there exists $\delta>0$, such that $T<\delta$, implies $\frac{1}{T}\,\|\mathbf{y}(0^+)\|\,\exp(\frac{-t}{T})<\epsilon^*$. Hence $\forall\overline{\epsilon}>0$, fixed $\epsilon^*<\overline{\epsilon}$, choose $T<\min\{\delta,\ \frac{\overline{\epsilon}-\epsilon^*}{M_{\dot{\mathbf{y}}}}\}$, then $\|\epsilon_1(t)\|<T\,M_{\dot{\mathbf{y}}}+\epsilon^*<\overline{\epsilon},\ \forall t>0$. Hence $\lim_{T\to 0}\|\epsilon_1(t)\|=0,\ \forall t>0$. Assume that $\lim_{T\to 0}\|\epsilon_k(t)\|=0,\ \forall t>0$, where k is chosen in the open indices set $(1,\ldots,l_p)$. Since $\psi_{k+1}(s)=\frac{s^{k+1}}{(1+Ts)^{k+1}}\,\mathbf{Y}(s)=\frac{s}{(1+Ts)}\,\psi_k(s)$, and $s\,\psi_k(s)=s\,\mathbf{Y}^{(k)}(s)-s\,\epsilon_k(s)=\mathbf{Y}^{(k+1)}(s)+\mathbf{y}^{(k)}(0^+)-s\,\epsilon_k(s)$, where $\mathbf{Y}^{(k+1)}(s)$ denotes the Laplace transform of the $(k+1)-th$ output derivative, it follows that:

$$\begin{aligned}\psi_{k+1}(s) &= \frac{s}{(1+Ts)}\,\psi_k(s)=\frac{1}{(1+Ts)}\,(\mathbf{Y}^{(k+1)}(s)\\ &+ \mathbf{y}^{(k)}(0^+)-s\,\epsilon_k(s)) \quad , \qquad (18)\end{aligned}$$

and

$$\begin{aligned}\epsilon_{k+1}(s)=\mathbf{Y}^{(k+1)}(s)-\psi_{k+1}(s)=T\,\frac{s}{(1+Ts)}\\ \mathbf{Y}^{(k+1)}(s)-\frac{\mathbf{y}^{(k)}(0^+)}{(1+Ts)}+\frac{s}{(1+Ts)}\epsilon_k(s) \qquad (19)\end{aligned}$$

the above expression, if $\mathbf{y}^{(k+1)}(t)$ is bounded and by using the induction assumption, immediately yields $\lim_{T\to 0}\|\epsilon_{k+1}(t)\|=0,\ \forall t>0$.

Let us consider the nonlinear system in Eq. (16). It easy to show that, if $\epsilon^*(t)=0,\ t\geq 0$, the origin $\mathbf{e}=0$ (of the unperturbed system) is locally asymptotically stable. In fact, consider the quadratic Lyapunov function candidate $V=\frac{1}{2}\mathbf{e}^T\,\mathbf{e}$, its time derivative results:

$$\begin{aligned}\dot{V} &= \mathbf{e}^T\,\dot{\mathbf{e}} \qquad (20)\\ &= \mathbf{e}^T\,[\left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\hat{\mathbf{x}},\mathbf{v}_0)\right)-\mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\,\left(\frac{\partial\Phi}{\partial\mathbf{x}}\,(\hat{\mathbf{x}},\mathbf{v})\right)]\,\mathbf{e}\\ &+ \mathbf{e}^T\,(\mathbf{p_f}(\mathbf{v}_0,\mathbf{e})+\mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\,\mathbf{p}_\Phi(\mathbf{v},\mathbf{e}))\\ &= -\mathbf{e}^T\,\mathbf{Q}\mathbf{e}+\mathbf{e}^T\,(\mathbf{p_f}(\mathbf{v}_0,\mathbf{e})+\mathbf{P}(\hat{\mathbf{x}},\mathbf{v})\,\mathbf{p}_\Phi(\mathbf{v},\mathbf{e})) \quad .\end{aligned}$$

Then, due to Eq. (17), there exists an open neighborhood of the trivial equilibrium in which $\dot{V}$ is negative definite. By applying the Lyapunov's direct method, the claim follows. Lemma 1 indicates that $\forall\epsilon_{\mathbf{e}}>0$ there exist $\delta_1>0$ and $K>0$ such that if the initial observation error is sufficiently small, i.e $\|\mathbf{e}(0)\|<\delta_1$ and choosing T sufficiently small such that $\|\epsilon_k(t)\|<\frac{K}{l_p},\ k=1,\ldots,l_p,\ t>0$, it follows $\|\epsilon^*\|<\sum_{k=1}^{l_p}\|\epsilon_k(t)\|<K$, and the observation error satisfies the condition $\|\mathbf{e}(t,0,\epsilon^*)\|<\epsilon_{\mathbf{e}},\ t>0$. ∎

Proof of Prop. 5.

Proof:

In the coordinates $(\mathbf{x}, \mathbf{e}) = (\mathbf{x}, \hat{\mathbf{x}} - \mathbf{x})$ the system (14) results:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{v}_0) \\
\dot{\mathbf{v}}_0 &= \mathbf{v}_1 \\
\dots & \\
\dot{\mathbf{v}}_{l_p-1} &= \overline{\alpha}(\mathbf{e} + \mathbf{x}, \mathbf{v}) \\
\mathbf{y}(\mathbf{x}) &= \mathbf{h}(\mathbf{x}) \\
\dot{\mathbf{e}} &= -\mathbf{Q}\mathbf{e} - \mathbf{P}(\hat{\mathbf{x}}, \mathbf{v})\epsilon^* + \mathbf{P}(\hat{\mathbf{x}}, \mathbf{v})\, \mathbf{p}_\Phi(\mathbf{v}, \mathbf{e}) \\
&- \mathbf{p}_\mathbf{f}(\mathbf{v}_0, \mathbf{e}) \quad . \qquad (21)
\end{aligned}
$$

As first step , we prove that if $\epsilon^* = 0$ the equilibrium of the unperturbed system (21) is asymptotically stable.

Let us denote, for convenience, $\mathbf{w} = [\mathbf{x}^T \;\; \mathbf{v}^T]^T \in \mathbf{X} \times \Re^{l_p m}$, $\hat{\mathbf{w}} = [\hat{\mathbf{x}}^T \;\; \mathbf{v}^T]^T$, $\overline{\alpha}(\mathbf{w}) = \overline{\alpha}(\mathbf{x}, \mathbf{v})$ and:

$$
\tilde{\mathbf{f}}(\mathbf{w}, \overline{\alpha}(\mathbf{w})) = \begin{pmatrix} \mathbf{f}(\mathbf{x}, \mathbf{v}_0) \\ \dot{\mathbf{v}}_0 \\ \dots \\ \overline{\alpha}(\mathbf{w}) \end{pmatrix} \quad , \qquad (22)
$$

notice also that $\hat{\mathbf{w}} - \mathbf{w} = [\mathbf{e}^T, \; 0^T]^T$ is only a function of the observation error $\mathbf{e}$. Since

$$
\tilde{\mathbf{f}}(\mathbf{w}, \overline{\alpha}(\hat{\mathbf{w}})) - \tilde{\mathbf{f}}(\mathbf{w}, \overline{\alpha}(\mathbf{w})) = \left(\frac{\partial \tilde{\mathbf{f}}}{\partial \overline{\alpha}}(\overline{\alpha}(\mathbf{w})) \right)
(\overline{\alpha}(\hat{\mathbf{w}}) - \overline{\alpha}(\mathbf{w})) + \mathbf{p}_{\tilde{\mathbf{f}}}(\mathbf{w}, \overline{\alpha}(\hat{\mathbf{w}}) - \overline{\alpha}(\mathbf{w})) \qquad (23)
$$

and

$$
\begin{aligned}
\overline{\alpha}(\hat{\mathbf{w}}) - \overline{\alpha}(\mathbf{w}) &= \left(\frac{\partial \overline{\alpha}}{\partial \mathbf{w}}(\mathbf{w}) \right) (\hat{\mathbf{w}} - \mathbf{w}) \\
&+ \mathbf{p}_{\overline{\alpha}}(\mathbf{w}, \mathbf{e}) \quad . \qquad (24)
\end{aligned}
$$

where the functions $p_{\tilde{\mathbf{f}}}(\mathbf{w}, \overline{\alpha}(\hat{\mathbf{w}}) - \overline{\alpha}(\mathbf{w}))$ and $\mathbf{p}_{\overline{\alpha}}(\mathbf{w}, \mathbf{e})$ vanish at $\mathbf{e} = 0$ with their first order partial derivatives, i.e. satisfy the conditions:

$$
\lim_{\mathbf{e} \to 0} \frac{p_{\tilde{\mathbf{f}}}(\mathbf{w}, \overline{\alpha}(\hat{\mathbf{w}}) - \overline{\alpha}(\mathbf{w}))}{\|\overline{\alpha}(\hat{\mathbf{w}}) - \overline{\alpha}(\mathbf{w})\|} = 0, \quad \lim_{\mathbf{e} \to 0} \frac{\mathbf{p}_{\overline{\alpha}}(\mathbf{w}, \mathbf{e})}{\|\mathbf{e}\|} = 0 \quad .
$$

Using Eqs. (23), (24) the closed-loop system (21) is written as:

$$
\begin{aligned}
\dot{\mathbf{w}} &= \tilde{\mathbf{f}}(\mathbf{w}, \overline{\alpha}(\mathbf{w})) + \tilde{\mathbf{p}}(\mathbf{w}, \mathbf{e}) \qquad (25) \\
\dot{\mathbf{e}} &= -\mathbf{Q}\mathbf{e} + \mathbf{P}(\hat{\mathbf{x}}, \mathbf{v})\, \mathbf{p}_\Phi(\mathbf{v}, \mathbf{e}) - \mathbf{p}_\mathbf{f}(\mathbf{v}_0, \mathbf{e}) \quad ,(26)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathbf{p}}(\mathbf{w}, \mathbf{e}) &= \left(\frac{\partial \tilde{\mathbf{f}}}{\partial \overline{\alpha}}(\overline{\alpha}(\mathbf{w})) \right) [\left(\frac{\partial \overline{\alpha}}{\partial \mathbf{w}}(\mathbf{w}) \right) (\hat{\mathbf{w}} - \mathbf{w}) \\
&+ \mathbf{p}_{\overline{\alpha}}(\mathbf{w}, \mathbf{e})] + \mathbf{p}_{\tilde{\mathbf{f}}}(\mathbf{w}, \overline{\alpha}(\hat{\mathbf{w}}) - \overline{\alpha}(\mathbf{w})) \qquad (27)
\end{aligned}
$$

is such that $\tilde{\mathbf{p}}(\mathbf{w}, 0) = 0, \; \forall \mathbf{w} \in \mathbf{X} \times \Re^{l_p m}$.

From Prop. 4, the system $\dot{\mathbf{w}} = \tilde{\mathbf{f}}(\mathbf{w}, \overline{\alpha}(\mathbf{w})) + \tilde{\mathbf{p}}(\mathbf{w}, 0)$ has an asymptotically stable equilibrium at $\mathbf{w} = 0$, since also $\dot{\mathbf{e}} = -\mathbf{Q}\mathbf{e} + \mathbf{P}(\hat{\mathbf{x}}, \mathbf{v})\, \mathbf{p}_\Phi(\mathbf{v}, \mathbf{e}) - \mathbf{p}_\mathbf{f}(\mathbf{v}_0, \mathbf{e})$ has an asymptotically stable equilibrium at $\mathbf{e} = 0$, then, as a consequence of a known property deriving from the center manifold theory [18], the cascade system (25), i.e. the unperturbed system ($\epsilon^* = 0$) in Eq. (14), has an asymptotically stable equilibrium at $(\mathbf{w}, \mathbf{e}) = (0, 0)$.

Consider the perturbed system:

$$
\begin{aligned}
\dot{\mathbf{w}} &= \tilde{\mathbf{f}}(\mathbf{w}, \overline{\alpha}(\mathbf{w})) + \tilde{\mathbf{p}}(\mathbf{w}, \mathbf{e}) \\
\dot{\mathbf{e}} &= -\mathbf{Q}\mathbf{e} - \mathbf{P}(\hat{\mathbf{x}}, \mathbf{v})\epsilon^* \\
&+ \mathbf{P}(\hat{\mathbf{x}}, \mathbf{v})\, \mathbf{p}_\Phi(\mathbf{v}, \mathbf{e}) - \mathbf{p}_\mathbf{f}(\mathbf{v}_0, \mathbf{e}) \qquad (28)
\end{aligned}
$$

$\forall \epsilon > 0$ there exist $\delta_1 > 0$ and $K > 0$ such that if the initial state is sufficiently small, i.e $\|[\mathbf{x}(0)^T, \mathbf{e}(0)^T]^T\| < \delta_1$, then since from Lemma 2 there exists $T > 0$ sufficiently small such that $\|\epsilon^*(0)\| < K, \; t > 0$, from Lemma 1, the solutions of (28) satisfy the condition $\|[\mathbf{x}^T(t, 0, \epsilon^*), \; \mathbf{e}^T(t, 0, \epsilon^*)]^T\| < \epsilon, \; t > 0$. ■

4 Redundant Observer Design

We now introduce, as quality measure of the current state estimation, the singularity function $s : \mathbf{X} \times \Re^{l_p m} \to \Re_+$, defined as $s(\mathbf{x}, \mathbf{v}) = \sqrt{\det\,(J(\mathbf{x}, \mathbf{v})^T J(\mathbf{x}, \mathbf{v}))}$. If assumption 1 is satisfied, than $s(\mathbf{x}, \mathbf{v}) > 0$ in the open neighborhood $\mathbf{X}^0 \times \mathbf{V}^0$. However, low values of $s(\mathbf{x}, \mathbf{v})$ indicate bad conditioned estimation.

In the following the Assumption 1 which is instrumental in the design of the observer (15), and in the local separation property (see Prop 5) is relaxed in the sense that we will allow the presence of bad inputs. Let us begin with the following definition.

Definition 2 *A real analytic function* $\mathbf{u}_b(t), \; t \geq t_0 \geq 0$ *is a bad input for the system (1) with respect to the initial condition* $\mathbf{x}(t_0)$*, if there exists, at least, an instant time* $\bar{t} \geq t_0$ *such that rank*$\left(\mathbf{J}(\mathbf{x}(\bar{t}), \mathbf{v}_b(\bar{t}))\right) < n$*, where* $\mathbf{x}(t) = \mathbf{x}(t, t_0, \mathbf{x}(t_0), \mathbf{u}_b)$ *is the flow associated to the system (1), and* $\mathbf{v}_b(\bar{t}) = [\mathbf{u}_b^T(\bar{t}), \dots, \mathbf{u}_b^{(l_p-1)^T}(\bar{t})]^T$.

Let us consider a bad input function $\mathbf{u}_b(t), \; t \geq t_0 \geq 0$, hence the singularity function defined as $s(\mathbf{x}, \mathbf{v}) = \sqrt{\det\,(J(\mathbf{x}, \mathbf{v})^T J(\mathbf{x}, \mathbf{v}))}$, at a certain time instant $\bar{t} \geq t_0$ satisfies the equation: $s(\mathbf{x}(\bar{t}), \mathbf{v}(\bar{t})) = 0$. We now give the following characterization to the set of all possible singularities.

Definition 3 *Denote with* $\mathbf{w} = [\mathbf{x}^T \;\; \mathbf{v}^T]^T$*, then if* $\left(\frac{\partial s}{\partial \mathbf{w}}(\mathbf{x}(\bar{t}), \mathbf{v}(\bar{t}))\right)$ *is nonzero the set* $S = \{(\mathbf{x}, \mathbf{v}) \in$

$\mathbf{X} \times \Re^{l_p m} : s(\mathbf{x}, \mathbf{v}) = 0\}$ *is a smooth manifold of dimension* $n + l_p m - 1$, *this is the singularity manifold associated to the system (1).*

The singular point $(\mathbf{x}(\bar{t}), \mathbf{v}(\bar{t}))$ is then an element of the singularity manifold S. Consider the equation:

$$\chi(\hat{\mathbf{x}}, \mathbf{v}) = \overline{\mathbf{P}}(\hat{\mathbf{x}}, \mathbf{v}) \mathbf{J}(\hat{\mathbf{x}}, \mathbf{v}) \quad , \tag{29}$$

where $\mathbf{J}(\hat{\mathbf{x}}, \mathbf{v})$ is the EOJ matrix, and $\chi \in \Re^{n \times n}$, $\overline{\mathbf{P}}(\hat{\mathbf{x}}, \mathbf{v}) \in \Re^{n \times l}$ have to be set. In the sequel, for simplicity notation, we will occasionally drop the dependence from $(\hat{\mathbf{x}}, \mathbf{v})$. In the Prop. 5, fixed $\chi = \mathbf{Q} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\hat{\mathbf{x}}, \mathbf{v}_0)\right)$, it was chosen $\overline{\mathbf{P}}(\hat{\mathbf{x}}, \mathbf{v}) = \mathbf{P}(\hat{\mathbf{x}}, \mathbf{v}) = \mathbf{J}^+(\hat{\mathbf{x}}, \mathbf{v})\chi$ to solve the Eq. (29). Let us define $l = (l_p + 1)\, p$, and denote with $\overline{\mathbf{P}}_i \in (\Re^l)^*$, $i = 1, \ldots n$ the rows of the matrix $\overline{\mathbf{P}}$, being $(\Re^l)^*$ the dual space of $\Re^l$. It is well-known that, if $l > n$, the matrix $\overline{\mathbf{P}} = \mathbf{P} = \mathbf{J}^+(\hat{\mathbf{x}}, \mathbf{v})\chi$ solves the problem:

$$\begin{cases} \min_{\overline{\mathbf{P}}} \sum_{i=1}^{n} \frac{1}{2} \overline{\mathbf{P}}_i\, \overline{\mathbf{P}}^T \\ \chi = \overline{\mathbf{P}} \mathbf{J}(\hat{\mathbf{x}}, \mathbf{u}) \quad . \end{cases} \tag{30}$$

Instead of minimize the norm of each row, we can choose to minimize the norm of the difference between $\overline{\mathbf{P}}_i$ and a suitable $\mathbf{P}_i^* \in (\Re^l)^*$, $i = 1, \ldots n$, to be chosen later. We consider the constrained minimization problem:

$$\begin{cases} \min_{\overline{\mathbf{P}}} \sum_{i=1}^{n} \frac{1}{2} (\overline{\mathbf{P}}_i - \mathbf{P}_i^*)\, (\overline{\mathbf{P}}_i - \mathbf{P}_i^*)^T \\ \chi = \overline{\mathbf{P}} \mathbf{J}(\hat{\mathbf{x}}, \mathbf{u}) \quad . \end{cases} \tag{31}$$

By introducing the Lagrangian multipliers $\lambda_i \in \Re^n$, $i = 1, \ldots, n$, the problem is solved by considering the function: $L(\overline{\mathbf{P}}, \mathbf{\Lambda}) = \sum_{i=1}^{n} [\frac{1}{2} (\overline{\mathbf{P}}_i - \mathbf{P}_i^*)\, (\overline{\mathbf{P}}_i - \mathbf{P}_i^*)^T + (\chi_i - \overline{\mathbf{P}}_i\, \mathbf{J})\, \lambda_i]$, where $\chi_i \in (\Re^n)^*$ are the rows of the matrix χ and $\mathbf{\Lambda} = [\lambda_1 \ \ldots \lambda_n] \in \Re^{n \times n}$ is the matrix of Lagrangian multipliers. Simple computations show that the conditions:

$$\begin{cases} \frac{\partial L(\overline{\mathbf{P}}, \mathbf{\Lambda})}{\partial \overline{\mathbf{P}}_i^T} = 0 \quad , \\ \frac{\partial L(\overline{\mathbf{P}}, \mathbf{\Lambda})}{\partial \lambda_i} = 0 \quad , \ i = 1, \ldots n \quad , \end{cases} \tag{32}$$

lead to the solution expressed in the matrix form: $\overline{\mathbf{P}} = \chi\, \mathbf{J}^+ + \mathbf{P}^* \left(\mathbf{I} - \mathbf{J}\, \mathbf{J}^+\right)$, where the matrix $\mathbf{P}^* \in \Re^{n \times l}$ is the collection of the covectors $\mathbf{P}_i^*$. It is simple to verify that Prop. 5 also holds if the above matrix $\overline{\mathbf{P}}$ (instead of $\mathbf{P}$) is chosen in the observer equation (15) with $\chi = \mathbf{Q} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\hat{\mathbf{x}}, \mathbf{v}_0)$, since $\mathbf{P}^* \left(\mathbf{I} - \mathbf{J}\, \mathbf{J}^+\right) \mathbf{J} = 0$, $\forall \mathbf{P}^* \in \Re^{n \times l}$. The matrix $\mathbf{P}^*$ can be chosen to deal with bad inputs during the state regulation.

Proposition 6 *Fixed an integer* l_p *such that* $(l_p + 1)\, p > n$, *by using the redundant observer* $\dot{\hat{\mathbf{x}}} = \dot{\hat{\mathbf{x}}}_o + \dot{\hat{\mathbf{x}}}_r$, *with* $\dot{\hat{\mathbf{x}}}_o = \mathbf{f}(\hat{\mathbf{x}}, \mathbf{v}_0) + \chi\, \mathbf{J}^+ \left(\hat{\xi} - \Phi(\hat{\mathbf{x}}, \mathbf{v})\right)$, *and* $\dot{\hat{\mathbf{x}}}_r = \mathbf{P}^* \left(\mathbf{I} - \mathbf{J}\, \mathbf{J}^+\right) \left(\hat{\xi} - \Phi(\hat{\mathbf{x}}, \mathbf{v})\right)$, *being* $\chi = \mathbf{Q} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\hat{\mathbf{x}}, \mathbf{v}_0)\right)$, *the Assumption 1 in the Prop. 5 can be substituted by the following conditions:*

- *chosen* $\mathbf{P}^* = r\, \Sigma$, $r \in \Re, \Sigma \in \Re^{n \times l}$, *along the flow of system (14), starting from* $\mathbf{x}(0)$, $\hat{\mathbf{x}}(0)$, $\mathbf{v}(0)$ *(the initial state of the high-pass filters is assumed zero), the condition*

$$\left(\frac{\partial s}{\partial \mathbf{x}}\, (\hat{\mathbf{x}}, \mathbf{v})\right) \Sigma\, (\mathbf{I} - \mathbf{J}\, \mathbf{J}^+)(\xi - \Phi(\hat{\mathbf{x}}, \mathbf{v})) \neq 0 \quad , \tag{33}$$

is satisfied;

- *the scalar* r *is computed as*

$$r = \frac{\dot{s}_{des} - \left(\frac{\partial s}{\partial \mathbf{v}}\, (\hat{\mathbf{x}}, \mathbf{v})\right) \dot{\mathbf{v}} - \left(\frac{\partial s}{\partial \mathbf{x}}\, (\hat{\mathbf{x}}, \mathbf{v})\right) \dot{\hat{\mathbf{x}}}_o}{\left(\frac{\partial s}{\partial \mathbf{x}}\, (\hat{\mathbf{x}}, \mathbf{v})\right) \Sigma\, (\mathbf{I} - \mathbf{J}\, \mathbf{J}^+)(\xi - \Phi(\hat{\mathbf{x}}, \mathbf{v}))} \quad , \tag{34}$$

where $\dot{s}_{des}(t)$ *is the desired singularity function derivative to be fixed.*

Proof: The time derivative of the singularity function $s(\hat{\mathbf{x}}, \mathbf{v})$ results:

$$\dot{s}(\hat{\mathbf{x}}, \mathbf{v}) = \left(\frac{\partial s}{\partial \mathbf{x}}(\hat{\mathbf{x}}, \mathbf{v})\right) \dot{\hat{\mathbf{x}}} + \left(\frac{\partial s}{\partial \mathbf{v}}(\hat{\mathbf{x}}, \mathbf{v})\right) \dot{\mathbf{v}} \quad . \tag{35}$$

which depends on the redundant observer $\dot{\hat{\mathbf{x}}} = \dot{\hat{\mathbf{x}}}_o + \dot{\hat{\mathbf{x}}}_r$. Assume that $\left(\frac{\partial s}{\partial \mathbf{x}}\, (\hat{\mathbf{x}}, \mathbf{v})\right) \Sigma\, (\mathbf{I} - \mathbf{J}\, \mathbf{J}^+)(\xi - \Phi(\hat{\mathbf{x}}, \mathbf{v}))$ does not vanish along the current flow of system (14), the scalar condition $\dot{s}(\hat{\mathbf{x}}, \mathbf{v}) = \dot{s}_{des}(t)$ is satisfied if Eq. (34) holds. If this is the case, defined the function $\tilde{s}(t) = s(\hat{\mathbf{x}}, (t)\mathbf{v}(t))$, the desired time derivative $\dot{s}_{des}(t)$, $t \geq 0$ can be chosen such that the integral $\tilde{s}(t) = \tilde{s}(0) + \int_0^t \dot{\tilde{s}}(\tau)\, d\tau$ does not vanish along the current flow. ∎

5 Example

Consider the model of an holonomic vehicle, which is able only to measure the distance from the origin of a priori fixed reference frame:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{u} \\ y(\mathbf{x}) &= \frac{1}{2}\, (x_1^2 + x_2^2) \quad , \end{aligned} \tag{36}$$

where $\mathbf{x} = [x_1,\ x_2]^T \in \mathbf{X}$, an open subset of $\Re^2$ containing the origin, is the state vector, $\mathbf{u} = [u_1,\ u_2]^T \in \Re^2$ is the control input vector, $y \in \Re$ is the output vector. We consider the problem of finding an output feedback control law which locally stabilizes the origin $\mathbf{x} = 0$. First notice that the first approximation of the above model around the trivial equilibrium is non–observable, but the observability rank condition is satisfied $\forall \mathbf{x} \in \mathbf{X}$. It follows that, since the model is analytic and weakly controllable, it is also weakly locally observable $\forall \mathbf{x} \in \mathbf{X}$. Fixed $l_p = 1$, the EOJ matrix results:

$$\mathbf{J}(\mathbf{x}, \mathbf{v}_0) = \begin{bmatrix} x_1 & x_2 \\ u_1 & u_2 \end{bmatrix} \quad , \tag{37}$$

the singularity manifold is given by the set $S = \{(\mathbf{x},\mathbf{u}) \in \mathbf{X} \times \Re^2 : \ x_1\,u_2 - x_2\,u_1 = 0\}$. This analysis indicates that the simple holonomic vehicle (36) is not uniformly observable [22, 18], in fact an input $\mathbf{u}_b(t)$, $t \geq t_0 \geq 0$, such that, at a certain time instant $\overline{t} \geq t_0$, satisfies the condition $x_1(\overline{t})\,u_2(\overline{t}) - x_2(\overline{t})\,u_1(\overline{t}) = 0$, i.e. the point $(\mathbf{x},\mathbf{u}_b)$ lies on the manifold S at time $\overline{t}$, causes a loss of rank in the EOJ matrix, the set of inputs $\mathbf{u}_b(t)$ which satisfy the above property are the bad inputs (see Definition 3). The assumption 1 is verified only in the region $(\mathbf{X} \times \Re^2) \setminus S$. However, by applying the framework described in Prop. 6, it is possible to avoid bad inputs during the output feedback. Moreover, the system can not globally transformed in an element of the class of nonlinear systems described in the recent reference [1]. In fact, the map:

$$
\begin{aligned}
\overline{x}_1(\mathbf{x},\mathbf{u}) &= y(\mathbf{x}) = \frac{1}{2}\,(x_1^2 + x_2^2) \\
\overline{x}_2(\mathbf{x},\mathbf{u}) &= \dot{y}(\mathbf{x}) = x_1\,u_1 + x_2\,u_2 \\
z_1(\mathbf{x},\mathbf{u}) &= u_1 \\
z_2(\mathbf{x},\mathbf{u}) &= u_2 \quad ,
\end{aligned} \tag{38}
$$

does not define a global diffeomorphism in the extended state space $\mathbf{X} \times \Re^2$.

5.0.1 Feedback stabilization with full observer:

We apply the observer design and the local separation property presented in Prop. 5, in the case $(l_p + 1)\,p = n = 2$, i.e. $l_p = 1$. This is is referred to as the *full observer* design. A stabilizing state feedback is simply $\mathbf{u} = \alpha_0(\mathbf{x}) = [-k\,x_1 \ \ -k\,x_2]^T$, $k > 0$. By using Prop. 4 with $V(\mathbf{x}) = \frac{1}{2}\,\mathbf{x}^T\,\mathbf{x}$, a stabilizing state feedback of the extended system:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{v}_0 \\
\dot{\mathbf{v}}_0 &= \nu \quad ,
\end{aligned} \tag{39}
$$

is $\nu = \overline{\alpha}(\mathbf{x},\mathbf{v}_0) = \dot{\alpha}_0(\mathbf{x},\mathbf{v}_0) - \mathbf{x} - \lambda\,(v_0 - \alpha_0(\mathbf{x})), \lambda > 0$, with $\dot{\alpha}_0(\mathbf{x},\mathbf{v}_0) = -k\,\mathbf{v}_0$. From Prop 5, the output feedback given by the equations

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \mathbf{v}_0 + \mathbf{P}(\hat{\mathbf{x}},\mathbf{v}_0)\,(\hat{\xi} - \Phi(\hat{\mathbf{x}},\mathbf{v}_0)) \\
\mathbf{P}(\hat{\mathbf{x}},\mathbf{v}_0) &= \mathbf{Q}\,\mathbf{J}^{+}(\hat{\mathbf{x}},\mathbf{v}_0) \\
\hat{\psi}_1(s) &= \frac{s}{(1 + T\,s)}\,\mathbf{Y}(s), \quad \psi_1(0) = 0 \\
\nu &= \overline{\alpha}(\hat{\mathbf{x}},\mathbf{v}_0)
\end{aligned} \tag{40}
$$

where $\mathbf{J}(\hat{\mathbf{x}},\mathbf{v}_0)$ is reported in (37), ensures practical stability of the trivial equilibrium of the nonlinear system (39), if assumption 1 is satisfied along the current system flow. In the reported trial, the control parameters are: $\mathbf{Q} = 0.5\,\mathbf{I}_2$, $T = 0.001$, $k = 1.5$, and $\lambda = 1$. The initial conditions of the real and estimated state are respectively $\mathbf{x} = [1, \ 1]^T$, and $\hat{\mathbf{x}} = [-2, \ 3]^T$. The results are shown in Fig. 1. Figs. 1.a) and 1.b) indicate the observer convergence and real state stabilization by the proposed output feedback. Fig. 1.c) and 1.d) report the plots of the added state variables $\mathbf{v}_0$ and of the control input $\nu = \overline{\alpha}(\hat{\mathbf{x}},\mathbf{v}_0)$. The singularity function, depicted in Fig. 1.e), indicates that the determinant of the EOJ matrix becomes small in the neighborhood of the equilibrium $(\mathbf{x}, \ \mathbf{v}_0) = (0, \ 0)$. Extensive simulations have shown a meaningful extension of the region of attraction.

5.0.2 Output feedback stabilization with redundant observer:

We referred to as *redundant observer*, the case $(l_p + 1)\,p > n = 2$. Fixed $l_p = 2$, the extended system results:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{v}_0 \\
\dot{\mathbf{v}}_0 &= \mathbf{v}_1 \\
\dot{\mathbf{v}}_1 &= \nu
\end{aligned} \tag{41}
$$

where $\mathbf{v}_0 = [v_{01}, \ v_{02}]^T \in \Re^2$, $\mathbf{v}_1 = [v_{11}, \ v_{12}]^T \in \Re^2$, and $\mathbf{v} = [\mathbf{v}_0^T, \ v_1^T]^T \in \Re^4$ are the added state variables. By applying Prop. 4 with $V(\mathbf{x}) = \frac{1}{2}\,\mathbf{x}^T\,\mathbf{x}$, to the extended system (41), simple calculations show that, fixed $\lambda_i > 0$, $i = 1, 2$ by choosing $\nu = \overline{\alpha}(\mathbf{x},\mathbf{v}_0,\mathbf{v}_1) = \dot{\alpha}_1 - (\mathbf{v}_0 - \alpha_0) - \lambda_2\,(\mathbf{v}_1 - \alpha_1)$ with $\alpha_1 = \dot{\alpha}_0 - \mathbf{x} - \lambda_1\,(\mathbf{v}_0 - \alpha_0(\mathbf{x}))$, the time derivative of $V_a(\mathbf{x}) = V(\mathbf{x}) + \frac{1}{2}\,((\mathbf{v}_0 - \alpha_0)^T\,(\mathbf{v}_0 - \alpha_0) + (\mathbf{v}_1 - \alpha_1)^T\,(\mathbf{v}_1 - \alpha_1))$ is negative definite, and the global asymptotic stability of the equilibrium $(\mathbf{x},\mathbf{v}) = (0, \ 0)$ follows from the Lyapunov's direct method. As second step, we apply the output feedback approach described in Props. 5, 6, in which $\chi = \mathbf{Q}$, the redundant EOJ matrix is

$$
\mathbf{J}(\hat{\mathbf{x}},\mathbf{v}_0,\mathbf{v}_1) = \begin{bmatrix} x_1 & x_2 \\ v_{01} & v_{02} \\ v_{11} & v_{12} \end{bmatrix} \quad , \tag{42}
$$

and, defined $r_d = (\frac{\partial s}{\partial \mathbf{x}}\,(\hat{\mathbf{x}},\mathbf{v}))\Sigma\,(\mathbf{I} - \mathbf{J}\,\mathbf{J}^{+})(\xi - \Phi(\hat{\mathbf{x}},\mathbf{v}))$, the scalar r is chosen as:

$$
r = \begin{cases} r^* & \text{if } |r_d| > 10^{-6} \text{ and } \log\,s(\hat{\mathbf{x}},\mathbf{v}) < 10^{-2} \\ 0 & \text{otherwise} \quad , \end{cases}
$$

where r^* is given in Eq. (34), $\dot{s}_{des}(t) = 0, \ t \geq 0$, and

$$
\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad .
$$

The output feedback control scheme ensures local practical stability of the trivial equilibrium of the nonlinear system (41), and avoid bad inputs during the state regulation. In the reported trial, the control parameters are: $\mathbf{Q} = \mathbf{I}_2$, $T = 0.001$, $k = 1.5$, $\lambda_1 = 1$, $\lambda_2 = 5$. The initial conditions of the real and estimated state are equal to the *full observer* case. The simulation results, shown in Fig. 2, indicates that the singularity function $s(\mathbf{x},\mathbf{v})$ remains around the value 10^{-2}, i.e. the threshold chosen in the r parameter design, while in the *full observer* case $s(\mathbf{x},\mathbf{v})$ decreased until about 10^{-6}.

6 Conclusions

A local separation property for locally observable analytic nonlinear systems has been shown. First, it it necessary to find a stabilizing state feedback of the extended system, where the number of the added state variables depend on the number of the output derivatives considered in the observer. The second step is to substitute the real state with the estimated one obtained from the proposed nonlinear observer in the control law. We have proven that the trivial equilibrium of the closed loop system remains locally practically stable. A strategy based on a redundant number of observables permits qto deal with observability matrix singularities, i.e. bad inputs during the state regulation. As future investigation, we are considering an extension of the presented framework to the output feedback stabilization in the large.

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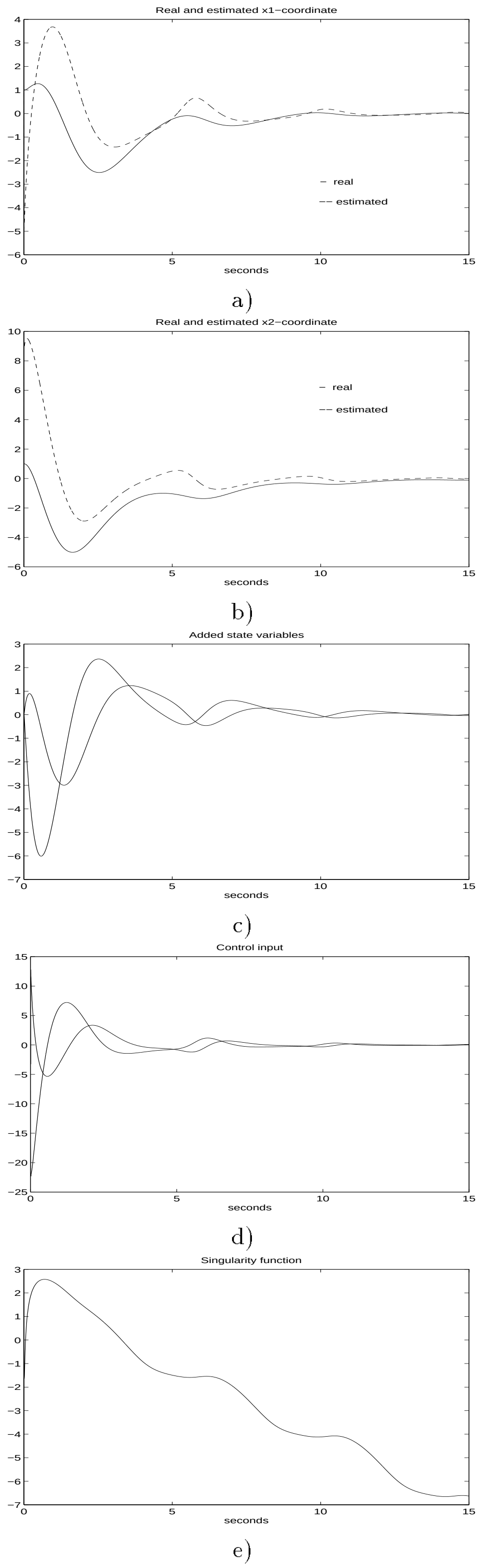

Figure 1: Example: output feedback stabilization with **full observer**. a) real and estimated x_1 component; b) real and estimated x_2 component; c) added state variables $\mathbf{v}_0$; d) control input $\overline{\alpha}(\mathbf{x}, \mathbf{v}_0)$; e) singularity index of the EOJ matrix: $\log s(\mathbf{x}, \mathbf{v}_0)$.

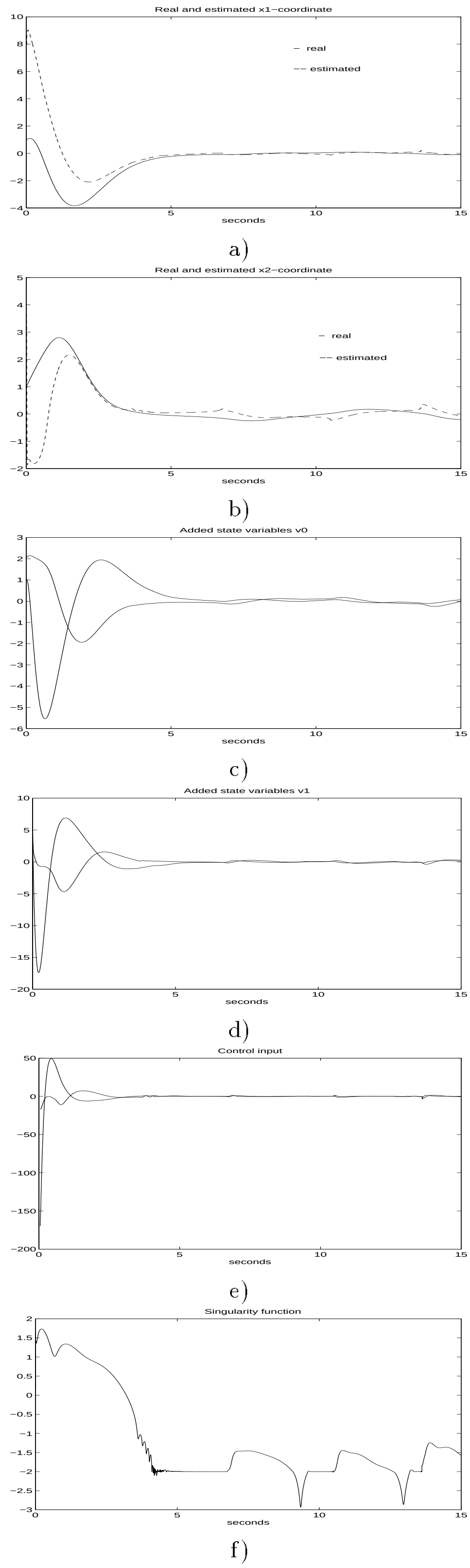

Figure 2: Example: output feedback stabilization with **redundant observer**. a) real and estimated x_1 component; b) real and estimated x_2 component; c) added state variables $\mathbf{v}_0$; d) added state variables $\mathbf{v}_1$; e) control input $\overline{\alpha}(\mathbf{x}, \mathbf{v}_0, v_1)$; f) singularity index of the EOJ matrix: $\log s(\mathbf{x}, \mathbf{v}_0, \mathbf{v}_1)$.