Necessary and Sufficient Conditions for Robust Perfect Tracking under Variable Structure Control*

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Abstract

The tracking control of linear MIMO systems with structured uncertainty is considered. A necessary and sufficient condition for robust asymptotic tracking employing variable structure techniques in the presence of multiplicative uncertainty is derived. The constructive proof of the theorem provides an explicit formula for controller synthesis.

1 Introduction

In this paper we investigate conditions under which a Variable Stucture Control (VSC) law of a standard type, which achieves asymptotic tracking on a linear plant $\mathbf{G}_D$, is guaranteed to accomplish the same performance on every plant in a given class of perturbed systems.

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We provide necessary and sufficient (Theorem 1) conditions for the existence of such VSC, and a formula for the explicit synthesis of the controller.

The problem is formulated precisely in Section 2. The new dominance conditions for decentralization and robustness are proposed in Section 3.

2 Problem formulation

Consider a family of m–inputs, m–outputs MIMO systems with multiplicative uncertainty, described in operator notation as

$$\mathbf{G} \; = \; \mathbf{G}_D(I + \Delta), \tag{1}$$

We assume that the nominal plant $\mathbf{G}_D$ is a strictly proper, n–th order linear operator, described by its impulse response matrix $\mathbf{G}_D(t)$ and transient response $\mathbf{g}^o(t)$ as

$$(\mathbf{G}_D \phi)(t) = \mathbf{G}_D(t) \star \phi(t) + \mathbf{g}^o(t),$$

where $\star$ denotes convolution. Uncertainty Δ is only supposed to be causal and $\mathbf{L}_\infty$ stable with finite gain ([1]). Denoting by $\mathbf{L}_q^\infty$ the space of functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}^q$ such that

$$\|\mathbf{f}(t)\|_\infty = \max_{k=1,q} \sup_{t \geq 0} |f_k(t)| < \infty,$$

such condition on Δ implies that, for all signals $\phi(t) \in \mathbf{L}_m^\infty$, there exist an $\mathbf{L}^\infty$–gain $\gamma_\Delta \in \mathbb{R}_+$ and a finite constant $Z^o \in \mathbb{R}_+$ such that

$$\|(\Delta\phi)(t)\|_\infty \leq \gamma_\Delta \;\; \|\phi(t)\|_\infty + Z^o.$$

Take a column–wise controllable canonical realization $\mathcal{S}$ of the nominal part $\mathbf{G}_D$,

$$\mathcal{S} : \left\{ \begin{array}{rcl} \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x} + \mathbf{B}\,(\mathbf{u} + \boldsymbol{\nu}), \quad \mathbf{x}(0) = \mathbf{x}^o \\ \mathbf{y} & = & \mathbf{C}\mathbf{x} \end{array} \right. . \tag{2}$$

where

$$\mathbf{A} = \; \mathsf{diag}\left(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(m)}\right), \quad \mathbf{B} = \; \mathsf{diag}\left(\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(m)}\right), \quad \mathbf{C} = \left[\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(m)}\right],$$

and $(\mathbf{A}^{(j)}, \mathbf{b}^{(j)}, \mathbf{C}^{(j)})$ are minimal realizations (of order $n^{(j)}$) in controllable canonical form of the j–th column of $\mathbf{G}_D$. Assume that initial conditions satisfy

$$\|\mathbf{x}^o\|_\infty \leq \rho \in \mathbb{R}_+. \tag{3}$$

Input disturbances $\boldsymbol{\nu}$ represent process noise satisfying the so–called *matching conditions* (cf. e.g. [2]). We assume $\boldsymbol{\nu} \in \mathbf{L}_m^\infty$, with

$$\|\boldsymbol{\nu}(t)\|_\infty \leq N \in \mathbb{R}_+. \tag{4}$$

Let the class of desired trajectories to be followed be described by the linear system (of order $n = \sum_{j=1}^m n^{(j)}$) $\mathcal{R}$,

$$\mathcal{R} : \begin{cases} \dot{\mathbf{r}} &= \mathbf{A}_r \mathbf{r} + \mathbf{B}_r \mathbf{v}_r, \quad \mathbf{r}(0) = \mathbf{r}^o \\ \mathbf{y}_r &= \mathbf{C}_r \mathbf{r} \end{cases}, \tag{5}$$

with $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$ in compatible column–wise controllable canonical form (hence $\mathbf{B}_r = \mathbf{B}$), $\mathbf{A}_r$ Hurwitz, $\mathbf{C}_r = \mathbf{C}$, and $\mathbf{v}_r \in \mathbf{L}_m^\infty$ with

$$\|\mathbf{v}_r(t)\|_\infty \leq V \in \mathbb{R}_+ \qquad \text{and} \qquad \|\mathbf{r}^o\|_\infty \leq \rho_r \in \mathbb{R}_+. \tag{6}$$

Restrictions on reference trajectories $\mathbf{y}_r$ amount to boundedness and some mild regularity conditions in case $\mathcal{S}$ is minimum–phase. If $\mathcal{S}$ has some zero in the closed right half–plane (CRHP), reference trajectories are generated through a system with the same CRHP zeroes. By this restriction on references perfect tracking is allowed also with nonminimum phase nominal systems. We assume in what follows that states of $\mathcal{S}$ are accessible to measurement, or that suitable observers are available (as e.g. in [3] and [4]). A combined observer/controller synthesis that applies directly to the present setup is described in [5].

2.1 Standard VSC design for MIMO linear plants

A standard technique for the synthesis of a VS controller for linear MIMO plants is succinctly reported below for reader's reference.

The dynamics of tracking error between reference states $\mathbf{r}$ and system states $\mathbf{x}$ (and hence, the dynamics of output tracking errors) can be chosen by enforcing a sliding motion on a linear manifold $\Sigma = \{\mathbf{x} \in \mathbb{R}^n | \boldsymbol{\sigma} = \mathbf{0}\}$, where $\boldsymbol{\sigma} \in \mathbb{R}^m$ is defined as

$$\boldsymbol{\sigma} \;=\; \boldsymbol{\Gamma}\left(\mathbf{x} \Leftrightarrow \mathbf{r}\right), \quad , \; \in \mathbb{R}^{m \times n}. \tag{7}$$

A convenient choice is $\boldsymbol{\Gamma} = \mathsf{diag}\left(\boldsymbol{\Gamma}^{(1)}, \ldots, \boldsymbol{\Gamma}^{(m)}\right)$, $\boldsymbol{\Gamma}^{(j)} \in \mathbb{R}^{1 \times n^{(j)}}$ such that $\boldsymbol{\Gamma}^{(j)} \mathbf{b}^{(j)} = 1$ (hence $\boldsymbol{\Gamma}\mathbf{B} = \mathbf{I}_m$). Pole assignment or LQ techniques can be employed for choosing the remaining $n^{(j)} \Leftrightarrow 1$ free parameters in $\boldsymbol{\Gamma}^{(j)}$, as described e.g. by Dorling and Zinober [6].

The dymanics of the state error $\mathbf{x} - \mathbf{r}$ can be obtained by the equivalent control method ([7]). The equivalent control is the input signal $\mathbf{u}_{eq}$ that solves $\dot{\boldsymbol{\sigma}} = \mathbf{0}$. We have

$$\mathbf{u}_{eq} \;=\; -\boldsymbol{\Gamma}(\mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\nu} - \dot{\mathbf{r}}). \tag{8}$$

Noticing that $(\mathbf{I} - \mathbf{B}\boldsymbol{\Gamma})\mathbf{A}_r = (\mathbf{I} - \mathbf{B}\boldsymbol{\Gamma})\mathbf{A}$, and using the form of the realizations of $\mathcal{S}$ and $\mathcal{R}$, the state error dynamics for the system restricted to the sliding surface Σ can be expressed as

$$\dot{\mathbf{x}} - \dot{\mathbf{r}} \;=\; (\mathbf{I} - \mathbf{B}\boldsymbol{\Gamma})(\mathbf{A}\mathbf{x} - \mathbf{A}_r\mathbf{r} - \mathbf{B}_r\mathbf{v}_r) \;=\; (\mathbf{I} - \mathbf{B}\boldsymbol{\Gamma})\mathbf{A}(\mathbf{x} - \mathbf{r}). \tag{9}$$

Note that only the coefficients of $\boldsymbol{\Gamma}$ actually appear in the sliding dynamics. Sliding motion on Σ yields the convergence of the states $\mathbf{x}$ to the states $\mathbf{r}$ with the dynamics imposed by the choice of $\boldsymbol{\Gamma}$. With a suitable choice of $\boldsymbol{\Gamma}$, then, outputs $\mathbf{y}$ during sliding asymptotically track reference ouputs $\mathbf{y}_{ri}$ under the equivalent control (8).

However, being the disturbance unknown, the equivalent control can not be synthesized directly. A common choice of the switching control law, which we will refer to as standard VSC design, consists of putting

$$\mathbf{u} \;=\; -\boldsymbol{\Gamma}(\mathbf{A}\mathbf{x} - \dot{\mathbf{r}}) - k\ \mathsf{sign}(\sigma), \tag{10}$$

with the $\mathsf{sign}(\cdot)$ function taken componentwise.

One says that a stable sliding regime exists on Σ if all system trajectories originating in a neighborhood of Σ point towards Σ, i.e. $\sigma^{(j)}\dot{\sigma}^{(j)} < 0$ for all components $\sigma^{(j)}$ of $\boldsymbol{\sigma}$. Such existence condition is met globally on the state space if and only if

$$k \;>\; \|\boldsymbol{\nu}(t)\|_{\infty}. \tag{11}$$

Furthermore, by choosing

$$k \;=\; N + \epsilon \tag{12}$$

where $\epsilon > 0$, it is guaranteed that the sliding manifold is reached in finite time, i.e. that $\boldsymbol{\sigma} = \mathbf{0}$ for all $t > \frac{\|\sigma(0)\|_{\infty}}{\epsilon}$.

In practical applications, it is very common that the plant is comprised of $N \leq m$, m_i–inputs, m_i–outputs weakly interacting square subsystems, with $m = m_1 + \ldots + m_N$. Correspondingly, the nominal plant has block–diagonal structure, $\mathbf{G}_D \;=\; \mathsf{diag}\,(\mathbf{G}_1, \ldots, \mathbf{G}_N)$.

The choice of a unique value of k for the switching part of inputs is possibly very conservative in this case. Smaller amplitude control signals can usually be obtained by decentralization, which consists in repeating the above synthesis procedure for each diagonal block $\mathbf{G}_i$ independently. In what follows, we refer to such multiple channel synthesis. The corresponding notation will differ from that introduced above only by a subscript referring to the input–output channel being considered (e.g., k_i will denote the amplitude of the switching control in channel i).

2.2 Problem statement

On these premises, we define the tracking performance of a standard VSC as follows

Definition 1 *A VSC law of the standard type (10) is said to achieve performance $\mathcal{P}_\Gamma^T$ on a system $\mathbf{G}$ if it ensures the establishment within time T, and the stability for all $t > T$, of a sliding regime, during which outputs of $\mathbf{G}$ asymptotically track reference trajectories (5), (6), with error dynamics determined by , $_i$, in spite of input disturbances as in (4).*

Furthermore, consider an $N \times N$ block partition of Δ in (1) conformal to that of $\mathbf{G}_D$, i.e. with blocks $\Delta_{ij} : \mathbf{L}^\infty_{m_j} \to \mathbf{L}^\infty_{m_i}$, and define $\mathbf{P} \in \mathbb{R}_+^{N\times N}$ as

$$\mathbf{P} = \{P_{ij}\} \qquad \text{with} \qquad P_{ij} \;=\; \gamma_{\Delta_{ij}} \;, \qquad \text{for} \qquad i,j = 1,\ldots,N. \tag{13}$$

We use matrix $\mathbf{P}$ to convey the information on the uncertainty structure by defining classes of disturbances $\mathcal{D}_P$ as

$$\mathcal{D}_P \;=\; \Big\{\Delta = \{\Delta_{ij}\} : \quad \Delta_{ij} \text{ causal, } \mathbf{L}_\infty \text{ stable with finite gain } \gamma_{\Delta_{ij}} \le P_{ij}\Big\} \;. \tag{14}$$

Explicitly, a small element P_{ij} in $\mathbf{P}$ indicates that all perturbations in $\mathcal{D}_P$ have blocks Δ_{ij} which are "small" in the above $\mathbf{L}^\infty$–gain sense. Formally, then

Definition 2 *A VSC law achieves performance $\mathcal{P}_\Gamma^T$ robustly with respect to $\mathcal{D}_P$ if it achieves $\mathcal{P}_\Gamma^T$ on $\mathbf{G} = \mathbf{G}_D(\mathbf{I} + \Delta)$, for all $\Delta \in \mathcal{D}_P$.*

The problem this paper is concerned with is the following:

Problem 1 *Given a nominal plant $\mathbf{G}_D$, a VS control law (10), and a class of structured multiplicative uncertainties $\mathcal{D}_P$, find conditions under which performance $\mathcal{P}_\Gamma^T$ is achieved robustly with respect to $\mathcal{D}_P$.*

3 Robustness Conditions

Consider the signal

$$\boldsymbol{\zeta}_i(t) \;=\; \sum_{j=1,N} \left(\Delta_{ij}(\mathbf{u}_j + \boldsymbol{\nu}_j)\right)(t)\ , \tag{15}$$

For, by design (10), $\mathbf{u}_i(t) \in \mathbf{L}^{\infty}_{m_i}$, $\boldsymbol{\zeta}_i(t) \in \mathbf{L}^{\infty}_{m}$. Notice that the perturbed plant outputs $\mathbf{y}_i$ can be written as

$$\mathbf{y}_i = \mathbf{G}_i * (\mathbf{u}_i + \boldsymbol{\nu}_i + \boldsymbol{\zeta}_i) + \mathbf{y}^o_i, \tag{16}$$

with $\mathbf{y}^o_i$ a transient term due to initial conditions of the nominal plant. A necessary and sufficient condition for robustness of VS control performance with respect to structured uncertain perturbations in the given class is given in the following theorem.

Theorem 1 *Given a nominal plant $\mathbf{G}_D$ and a class of structured multiplicative uncertainties $\mathcal{D}_P$, there exists a standard VSC law as in (10) achieving performance $\mathcal{P}^T_\Gamma$ robustly with respect to $\mathcal{D}_P$, if and only if for the Perron–Frobenius root of $\mathbf{P}$ it holds*

$$\rho_{PF}\left(\mathbf{P}\right) < 1. \tag{17}$$

Proof of sufficiency part.

Write the realization of system (16) under the VS control law (10) as

$$\left\{ \begin{array}{lcl} \dot{\mathbf{w}}_i & = & \mathbf{A}_i\mathbf{w}_i + \mathbf{B}_i(\mathbf{u}_i + \boldsymbol{\nu}_i + \boldsymbol{\zeta}_i), \quad \mathbf{w}_i(0) = \mathbf{x}^o_i \\ \mathbf{y}_i & = & \mathbf{C}_i\mathbf{w}_i \end{array} \right. ,$$

with $\mathbf{w}_i \in \mathbb{R}^{n_i}$, and

$$\mathbf{u}_i \;=\; \mathbf{u}'_i \Leftrightarrow k_i\ \mathsf{sign}(\boldsymbol{\Gamma}_i(\mathbf{w}_i \Leftrightarrow \mathbf{r}_i)) \tag{18}$$

with $\boldsymbol{\Gamma}_i \in \mathbb{R}^{m_i \times n_i}$ and

$$\mathbf{u}'_i \;=\; \Leftrightarrow \boldsymbol{\Gamma}_i(\mathbf{A}_i\mathbf{w}_i \Leftrightarrow \dot{\mathbf{r}}_i). \tag{19}$$

The existence of a stable sliding regime on Σ_i for $t \geq T$ is guaranteed if and only if it holds

$$k_i \;>\; \|\boldsymbol{\nu}_i(t+T) + \boldsymbol{\zeta}_i(t+T)\|_\infty\,. \tag{20}$$

Obviously, it holds $\|\boldsymbol{\nu}_i(t+T)\|_\infty \leq N_i$. Furthermore, since for all $\Delta \in \mathcal{D}_P$ $\gamma_{\Delta_{ij}} \leq P_{ij}$, from (15), we get

$$\|\boldsymbol{\zeta}_i(t+T)\|_\infty \;\leq\; \sum_{j=1,N} P_{ij}\left(\|\mathbf{u}_j(t+T)\|_\infty + \|\boldsymbol{\nu}_j(t+T)\|_\infty\right) + Z^o_{ij} \tag{21}$$

An upper bound on the control signal (10) is given by $\|\mathbf{u}_j(t+T)\|_\infty \le \left\|\mathbf{u}'_j(t+T)\right\|_\infty + k_j$. Consider the transformed state variables $\boldsymbol{\eta}_j = \mathbf{T}_j(\mathbf{x}_j \Leftrightarrow \mathbf{r}_j)$, with $\mathbf{T}_j \in \mathbb{R}^{n_j \times n_j}$ given by

$$\mathbf{T}_j = \begin{bmatrix} \mathbf{I}_{n_j - m_j} & \mathbf{0} \\ \boldsymbol{\Gamma}_j^1 & \boldsymbol{\Gamma}_j^2 \end{bmatrix} \mathbf{M}_j, \tag{22}$$

where $\mathbf{M}_j$ can be obtained (by QU decomposition of $\mathbf{B}_j$) such that

$$\mathbf{M}_j \mathbf{B}_j = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_j^2 \end{bmatrix}, \quad \boldsymbol{\Gamma}_j \mathbf{M}_j^{-1} = \begin{bmatrix} \boldsymbol{\Gamma}_j^1 & \boldsymbol{\Gamma}_j^2 \end{bmatrix}.$$

Notice that $\boldsymbol{\Gamma}_j^2$ and $\mathbf{B}_j^2$ are nonsingular (see e.g. [8]). Partitioning the transformed state $\boldsymbol{\eta}_j$ as

$$\boldsymbol{\eta}_j = \begin{bmatrix} \boldsymbol{\eta}_j^{(1)} \\ \boldsymbol{\eta}_j^{(2)} \end{bmatrix}, \qquad \boldsymbol{\eta}_j^{(1)} \in \mathbb{R}^{n_j - m_j},\ \boldsymbol{\eta}_j^{(2)} \in \mathbb{R}^{m_j}, \qquad \mathbf{T}_j \mathbf{A}_j \mathbf{T}_j^{-1} = \begin{bmatrix} \mathbf{A}_j^{11} & \mathbf{A}_j^{12} \\ \mathbf{A}_j^{21} & \mathbf{A}_j^{22} \end{bmatrix},$$

the sliding regime condition $\boldsymbol{\sigma}_i = \mathbf{0}$ is rewritten as $\boldsymbol{\eta}_j^{(2)} = 0$, while the evolution in the $(n_j \Leftrightarrow m_j)$ reduced state space is described by

$$\boldsymbol{\eta}_j^{(1)}(t) = \exp\left(\mathbf{A}_j^{11}(t \Leftrightarrow T)\right)\mathbf{T}_j(\mathbf{x}_j(T) \Leftrightarrow \mathbf{r}_j(T)) .$$

By means of (19) and (5), signal $\mathbf{u}'_j$ can be rewritten as

$$\mathbf{u}'_j = \Leftrightarrow \boldsymbol{\Gamma}_j \mathbf{A}_j \mathbf{T}_j^{-1} \boldsymbol{\eta}_j \Leftrightarrow (\boldsymbol{\Gamma}_j(\mathbf{A}_j \Leftrightarrow \mathbf{A}_{rj})\exp(\mathbf{A}_{rj}t)\mathbf{B}_{rj} \Leftrightarrow \delta(t)\mathbf{I}) * \mathbf{v}_{rj} . \tag{23}$$

Since $\mathbf{A}_j^{11}$ is Hurwitz (this proceeds from the choice of stable tracking dynamics in $\boldsymbol{\Gamma}_j$), for any $\mathbf{x}_j(T)$ such that $\|\mathbf{x}_j(T) \Leftrightarrow \mathbf{r}_j(T)\|_\infty \le \rho_j^s \in \mathbb{R}_+$, control (19) is in $\mathbf{L}_{m_j}^\infty$ and $\left\|\mathbf{u}'_j(t+T)\right\|_\infty$ can be bounded by

$$\begin{aligned} U_j = {} & \left\|\boldsymbol{\Gamma}_j \mathbf{A}_j \mathbf{T}_j^{-1}\right\|_\infty \alpha_{\mathbf{A}_j^{11}} \|\mathbf{T}_j\|_\infty \rho_j^s + \|\boldsymbol{\Gamma}_j(\mathbf{A}_j \Leftrightarrow \mathbf{A}_{rj})\|_\infty \alpha_{\mathbf{A}_{rj}} \rho_{rj} \\ & + \|\boldsymbol{\Gamma}_j(\mathbf{A}_j \Leftrightarrow \mathbf{A}_{rj}) \exp\left(\mathbf{A}_{rj}t\right)\mathbf{B}_{rj} \Leftrightarrow \delta(t)\mathbf{I}\|_{\mathcal{A}} V_j , \end{aligned} \tag{24}$$

where $\alpha : \mathbb{R}^{m \times m} \to \{\mathbb{R}_+, \infty\}$, $\alpha_{\mathbf{M}} = \sup_{t \ge 0} \|\exp(\mathbf{M}t)\|_\infty$. Efficient techniques for providing such bounds of matrix exponentials can be found e.g. in [9] and [10].

Recapitulating, bounds on the peak norm of the vector of equivalent input disturbances $\boldsymbol{\zeta}_i(\cdot)$ after time T are provided as

$$\|\boldsymbol{\zeta}_i(t+T)\|_\infty \le \sum_{j=1,N} P_{ij}(U_j + k_j + N_j) + Z_i^o, \tag{25}$$

with $Z_i^o = \sum_{j=1,N} Z_{ij}^o$. Introducing the vector notation

$$\mathbf{z} = [\|\boldsymbol{\zeta}_1(t+T)\|_\infty, \cdots, \|\boldsymbol{\zeta}_N(t+T)\|_\infty]^T; \quad \mathbf{u} = [U_1, \cdots, U_N]^T,$$

$$\mathbf{k} = [k_1, \cdots, k_N]^T; \quad \mathbf{n} = [N_1, \cdots, N_N]^T; \quad \mathbf{z}^o = [Z_1^o, \cdots, Z_N^o]^T;$$

inequalities (25) are rewritten as

$$\mathbf{z} \ \leq \ \mathbf{P}(\mathbf{k} + \mathbf{u} + \mathbf{n}) + \mathbf{z}^o \tag{26}$$

(inequality signs in vectorial relations are meant elementwise). Accordingly, condition (20) is verified provided that

$$\mathbf{k} \ > \ \mathbf{Pk} + (\mathbf{I} + \mathbf{P})\mathbf{n} + \mathbf{Pu} + \mathbf{z}^o \ \geq \ \mathbf{n} + \mathbf{z}. \tag{27}$$

Introducing $\boldsymbol{\epsilon} = [\epsilon_1, \cdots, \epsilon_N]^T$, the VSC law (10) with $\mathbf{k} = \mathbf{n} + \boldsymbol{\epsilon}$ guarantees the existence of a sliding regime yielding performance $\mathcal{P}_\Gamma$ on $\mathbf{G}$, provided that

$$\boldsymbol{\epsilon} \ > \ \mathbf{P}\boldsymbol{\epsilon} + \mathbf{P}(\mathbf{u} + 2\mathbf{n}) + \mathbf{z}^o,$$

or, equivalently,

$$\boldsymbol{\epsilon} \ = \ \mathbf{P}\boldsymbol{\epsilon} + \mathbf{P}(\mathbf{u} + 2\mathbf{n}) + \mathbf{z}^o + \boldsymbol{\beta}, \quad \boldsymbol{\beta} > 0\ . \tag{28}$$

From the theory of positive matrices (see e.g. [11]), a nonnegative solution $\boldsymbol{\epsilon}$ to this equation exists for nonnegative $\mathbf{P}$,$\mathbf{n}$,$\mathbf{u}$, and $\mathbf{z}^o$, if and only if the Perron–Frobenius root of $\mathbf{P}$ is smaller than 1.

Under this hypothesis, the VS controller (10) is completely defined by the choice of parameters $\mathbf{k}$ in the set

$$\mathbf{k} = \mathbf{n} + (\mathbf{I} \Leftrightarrow \mathbf{P})^{-1} (2\mathbf{Pn} + \mathbf{Pu} + \mathbf{z}^o + \boldsymbol{\beta}), \quad \boldsymbol{\beta} > 0\ . \tag{29}$$

Such set is a cone in $\mathbb{R}_+^N$ with vertex in $\mathbf{n} + (\mathbf{I} \Leftrightarrow \mathbf{P})^{-1}(2\mathbf{Pn} + \mathbf{Pu} + \mathbf{z}^o)$ and positively spanned by the columns of $(\mathbf{I} \Leftrightarrow \mathbf{P})^{-1}$. In order to guarantee that sliding regimes are established by time T for all channels, it will suffice to pick

$$\boldsymbol{\beta} \ \text{ such that } \epsilon_i \geq \|\boldsymbol{\sigma}_i(0)\|_\infty / T, \ \forall i. \tag{30}$$

The existence of such $\boldsymbol{\beta}$ is guaranteed by the fact that $(\mathbf{I} \Leftrightarrow \mathbf{P})$ is a full rank matrix. Q.E.D.

Proof of necessity part. We need to show that, if $\rho_{PF}(\mathbf{P}) \geq 1$, there exist some $\Delta \in \mathcal{D}_P$ and some reference inputs $\mathbf{v}_{ri}$, and input noise $\boldsymbol{\nu}_i$, such that conditions (20) for a stable sliding regime are violated for some i.

In fact, take for simplicity $Z^o_{ij} = 0$ and $\boldsymbol{\nu}_i = \mathbf{0}$ for all channels, and identically null reference states $\mathbf{r}_i(t) \equiv \mathbf{0}$, for all $i = 1, \ldots, N$. From (15) and (19), we get

$$\boldsymbol{\zeta}_i(t) \;=\; \Leftrightarrow \sum_{j=1,N} \Delta_{ij}(\boldsymbol{\Gamma}_j \mathbf{A}_j \mathbf{z}_j + k_j \mathsf{sign}(\boldsymbol{\sigma}_j))(t) \; .$$

A particular element in the class $\mathcal{D}_P$ can be always chosen as a block–partitioned matrix whose i,j–th block $\boldsymbol{\Delta}_{ij}$ is built such that

$$\mathbf{f}(t) = \Big(f_1(t), \cdots, f_{m_j}\Big) \in \mathbf{L}^\infty_{m_j} \to (\Delta_{ij}\mathbf{f})(t) = (P_{ij} f_1(t), \cdots, P_{ij} f_1(t)) \in \mathbf{L}^\infty_{m_i}.$$

By this choice, and considering at time t^* a perturbation of the sliding regime in a vicinity of the Σ_j manifolds such that

$$\boldsymbol{\sigma}_j = \boldsymbol{\Gamma}_j \mathbf{z}_j < \mathbf{0}$$

and

$$\boldsymbol{\Gamma}_j \mathbf{A}_j \mathbf{z}_j \leq \mathbf{0}$$

for all j, we have

$$\|\boldsymbol{\zeta}_i(t^*)\|_\infty = \sum_{j=1,N} P_{ij} k_j + \eta_j \; ,$$

where $\eta_j = \left\| \Leftrightarrow \sum_{j=1,N} P_{ij} \boldsymbol{\Gamma}_j \mathbf{A}_j \mathbf{z}_j \right\|_\infty \geq \mathbf{0}$. Necessary and sufficient conditions (20) are then expressed in vector notation as $\mathbf{k} \geq \mathbf{P}\mathbf{k} + \boldsymbol{\eta}$, with $\boldsymbol{\eta} = [\eta_1, \cdots, \eta_N]$. The latter inequality, by the Perron–Frobenius theorem, can not be satisfied by any positive $\mathbf{k}$ if $\rho_{PF}(\mathbf{P}) \geq 1$. Q.E.D.

4 Discussion

Remark 1. The robust performance condition $\rho_{PF}(\mathbf{P}) < 1$ does not depend on any parameter of the plant or of the controller other than the structured disturbance bounds in $\mathbf{P}$. Different specifications of $\mathcal{P}^T_\Gamma$, by changing the sliding regime onset time T or the sliding dynamics , , would affect the actual value of the switching part of the control signals. If bounds on available controls are present in the problem setup, the existence of a VS control design is equivalent to the existence of an intersection between the cone (29), the regions defined by (30) and by control bound surfaces (typically, a convex programming problem).

Remark 2. Conditions equivalent to (17) can be obtained from the theory of nonnegative and M–matrices as

- there exists an induced norm $\|\cdot\|$ on $\mathbb{R}^{N\times N}$ such that $\|\mathbf{P}\| < 1$;
- $\mathbf{W} = \mathbf{I} \Leftrightarrow \mathbf{P}$ is an M–matrix,

Furthermore, easy–to–check sufficient conditions for (17) to be met are derived from Gershgorin's theorem as

$$\|\mathbf{P}\|_\infty < 1 \qquad\qquad \|\mathbf{P}\|_1 < 1 \tag{31}$$

i.e., in terms of conventional row or column dominance. Note also that, according to the theory of generalized diagonal dominance (see e.g. [12]), conditions in Theorem 1 guarantee the existence of an input–output scaling matrix § with positive elements such that $§^{-1}$ $\mathbf{P}§$ satisfies one of the (31).

Remark 3. Condition (17) is related to well–known quasi–block diagonal dominance conditions ([13], [14]). The latter have been traditionally formulated, in the hypothesis that uncertainties are linear and time–invariant, such that blocks $\mathbf{\Delta}_{ij}$ can be described by their transfer function matrix $\mathbf{\Delta}_{ij}(s)$, as

$$\rho\left(\left\{(\mathbf{K}_{ii}^{-1} + \mathbf{G}_i)^{-1}\mathbf{G}_i\mathbf{\Delta}_{ij}(s)\right\}\right) \;<\; 1, \;\; \forall s \in \mathcal{D}; \tag{32}$$

$$\rho_{PF}\;\left(\left\{\left\|(\mathbf{K}_{ii}^{-1} + \mathbf{G}_i)^{-1}\mathbf{G}_i\mathbf{\Delta}_{ij}(s)\right\|\right\}\right) \;<\; 1, \;\; \forall s \in \mathcal{D}, \tag{33}$$

where $\mathcal{D}$ is the Nyquist contour, $\rho(\cdot)$ is the spectral radius of a matrix on the complex field, and $\|\cdot\|$ is any induced norm on the space of complex matrices of given dimensions. In the interesting limit case that dominance is seeked for high gains $\mathbf{K}_{ii}$ that enforce arbitrary small tracking errors on minimum–phase nominal systems, in fact, from (33) one gets

$$\lim_{\|\mathbf{K}\|\to\infty} \rho_{PF}\;\left(\left\{\left\|(\mathbf{K}_{ii}^{-1} + \mathbf{G}_i)^{-1}\mathbf{G}_i\mathbf{\Delta}_{ij}(s)\right\|\right\}\right) \;<\; 1 \;\Leftrightarrow\; \rho_{PF}\left(\{\|\mathbf{\Delta}_{ij}(s)\|\}\right) \;<\; 1, \;\; \forall s \in \mathcal{D}; \tag{34}$$

For linear time–invariant uncertainty, the $\mathbf{L}^\infty$–gain of blocks $\mathbf{\Delta}_{ij}$ can be replaced by the $\mathcal{A}$–norm of their impulse response matrices, denoted by $\|\mathbf{\Delta}_{ij}(t)\|_{\mathcal{A}}$, such that condition

(17) specializes in this case as

$$\rho_{PF}\left(\left\{\|\mathbf{\Delta}_{ij}(t)\|_{\mathcal{A}}\right\}\right) \ < \ 1. \tag{35}$$

It follows (see e.g. [1]) that condition (35) is stricter than (34) in general. This could be expected, as condition (34) only guarantees robust stability, while (35) achieves robust performance.

Remark 4. An important result of Khammash [15] presents necessary and sufficient conditions for robust steady–state tracking in a linear time–invariant system in the presence of linear, time–varying, norm–bounded, structured perturbations. The relation to the above theorem 1, is interesting to discuss, especially in view of the resemblance of the criteria (both are given in terms of the Perron–Frobenius root of a matrix of $\mathcal{A}$–norms of impulse response matrices, see remark 3).

The essential difference between Khammash's framework and the present one is in the type of control action which is assumed. While in Khammash's work linear controls with finite gain are employed, the usage of switching control signals avails VSC techniques with controls of bounded amplitude, but infinite gain. One of the consequences of this fact is that we were able to provide necessary and sufficient conditions for the robust asymptotic tracking of arbitrary references (to within specification (5)), while the Internal Model Principle allows any finite–gain controller to enforce asymptotic tracking of only a finite number of reference signals. This fact makes direct comparisons of the two methods impossible. However, if we investigate by Khammash's method under what conditions it is possible to obtain arbitrarily small steady state errors by increasing the controller gains, the same condition $\rho_{PF}(\mathbf{P}) < 1$ is obtained. Ours is not a particular case of Khammash's work, however, since his results only apply to linear plant/controller systems. Moreover, our method explicitly provides a thorough controller synthesis procedure.

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