Embodying Desired Behavior in Variable Stiffness Actuators ⋆

Ludo C. Visser * **Stefano Stramigioli** * **Antonio Bicchi** **

* *Department of Electrical Engineering, University of Twente, The Netherlands*
{l.c.visser,s.stramigioli}@utwente.nl
** *Centro Interdipartimentale di Ricerca "E. Piaggio", University of Pisa, Italy*
bicchi@centropiaggio.unipi.it

Abstract: Variable stiffness actuators are a class of actuators with the capability of changing their apparent output stiffness independently from the actuator output position. This is achieved by introducing internally a number of compliant elements, and internal actuated degrees of freedom that determine how these compliant elements are perceived at the actuator output. The introduction of a mechanical compliance introduces intrinsic, passive oscillatory behavior to the system, but rather than trying to minimize this effect, the question arises if it can be exploited for the actuation of periodic motions. In this work, we propose a strategy to control the variable stiffness actuator optimally, with respect to a cost criterion, to a desired periodic motion of the output. In particular, the cost criterion provides a measure of embodiment of the desired behavior in the passive behavior of the variable stiffness actuator, i.e., the variable stiffness actuator is controlled such that its passive behavior is as close as possible to the desired behavior and thus that the control effort is minimized.

Keywords: Variable stiffness actuators, Mathematical models, Nonlinear analysis, Optimization problems, Periodic motion, Robotics

1. INTRODUCTION

Variable stiffness actuators are capable of changing the apparent output stiffness independently from the output position. This is achieved by introducing one or more internal elastic elements to the actuator, and a number of actuated internal degrees of freedom that determine how the elastic elements are sensed at the output. In this way, a mechanical compliance with variable stiffness is introduced, that decouples the actuated joint from the actuator itself (Bicchi and Tonietti, 2004). In many emerging robotic applications, such as walking robots, service and rehabilitation robotics, and prostheses and orthoses, physical human-robot and robot-environment interaction is an integral part, and in these cases the introduction of the mechanical compliance guarantees an intrinsic level of safety and stability. It can be seen as a mechanical implementation of impedance control (Hogan, 1985).

In robotic applications where the motions are mostly periodic, the introduction of a mechanical compliance allows a temporary storage of energy when negative work is done by the actuator (Stramigioli et al., 2008; van Dijk and Stramigioli, 2008). By observing that the added mechanical compliance introduces an oscillatory, passive behavior to the system, it was shown by Uemura and Kawamura (2009) that by tuning the stiffness properly to the desired motion, more energy efficient actuation of periodic motions can be achieved. The optimal stiffness was assumed to be constant, so that any costs related to changing this stiffness only become relevant when the desired motion is changed.

In this work, we show that the behavior of a variable stiffness actuator can be accurately described by the behavior of a spring with variable stiffness and equilibrium position. A cost is associated to changing the equilibrium position and the stiffness, both in terms of the deviation from the initial position and the rate of change. Then, by allowing both the equilibrium position and the stiffness to change dynamically, the desired output motion of the actuator output can be achieved optimally with respect to this cost. The cost provides a measure of embodiment of desired behavior in the passive dynamics of the variable stiffness actuator: if the deviations from the initial conditions are minimal, then the passive dynamics that the system would show when the equilibrium position and stiffness are not changed, already approaches the desired behavior as close as possible. The optimal initial conditions can be found that minimize the cost criterion.

The paper is organized as follows. Section 2 describes the generalized behavior of variable stiffness actuators, with the aim of rendering our approach independent from specific actuator designs. Then, in Section 3, the problem is formally stated and explained in detail. In Section 4, a nominal solution to the problem is provided, which is then optimized according to the cost criterion in Section 5. The

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effectiveness of our approach is illustrated by algebraic and simulation examples in Section 6. Concluding remarks and an outline for future work is given in Section 7.

2. GENERALIZED BEHAVIOR OF VARIABLE STIFFNESS ACTUATORS

In this Section, we present a port-Hamiltonian model for variable stiffness actuators. This model is an extension of the model presented by Visser et al. (2010). Furthermore, we propose a change of coordinates to capture the behavior of the variable stiffness actuator, irrespective of the particular actuator design.

2.1 Generic Port-Hamiltonian Model of Variable Stiffness Actuators

A generic port-based model for variable stiffness actuators was introduced by Visser et al. (2010), in which it was assumed that:

- the variable stiffness actuator has a number of internal elastic elements, described by a state s and an energy function $H_s(s)$ describing the storage of elastic energy;
- there are a number of actuated internal degrees of freedom, with configuration variables q;
- the behavior at the actuator output, with one degree of freedom, is determined by the intrinsic properties of the elastic elements and the configuration of the internal degrees of freedom.

Moreover, since the model aims to capture the working principle of the variable stiffness actuator, internal inertias and friction were not incorporated in the model. Under these assumptions, it was shown that the actuator behavior can be accurately described in a port-based setting by:

$$\begin{bmatrix} \dot{s} \\ \tau_q \\ \tau_r \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & A(q,r) & B(q,r) \\ -A^T(q,r) & 0 & 0 \\ -B^T(q,r) & 0 & 0 \end{bmatrix}}_{D_0(q,r)} \begin{bmatrix} \frac{\partial H_s}{\partial s} \\ \dot{q} \\ \dot{r} \end{bmatrix} \tag{1}$$

where the skew-symmetric matrix $D_0(q,r)$ describes the power continuous port interconnection. In particular, a *storage port*, an *output port* and a *control port* can be identified. The storage port is described by the power conjugate pair $(\dot{s}, \frac{\partial H_s}{\partial s})$, where $\dot{s}$ denotes the rate of change of the state s of the elastic elements and $\frac{\partial H_s}{\partial s}$ denotes the force generated by these elements. The output port is described by the pair $(\dot{r}, \tau_r)$, with $\dot{r}$ the rate of change of the output position r, and τ_r the colocated force. The pair $(\dot{q}, \tau_q)$ describes the control port, where $\dot{q}$ denotes the rate of change of the configuration variables q, and τ_q are the generalized collocated forces. The matrices $A(q,r)$ and $B(q,r)$ are the algebraic Jacobians of the kinematic relation $\lambda : (q,r) \mapsto s$, that relates the actuator output position r and the configuration q of the internal degrees of freedom to the state s of the internal elastic elements. In particular:

$$A(q,r) := \frac{\partial \lambda}{\partial q}, \quad B(q,r) := \frac{\partial \lambda}{\partial r}$$

The description (1) assumes ideal internal actuators, and thus velocity control of $\dot{q}$. In practice however, the internal actuators have an inertia, and the torque required to achieve a certain desired $\dot{q}$ follows from the formulation of an appropriate control law. Extending (1) to include these internal inertias is straightforward. Letting $M = \text{diag}(m_1, \ldots, m_n)$ denote the constant inertia matrix, we then obtain the following Hamiltonian energy function:

$$H(\rho, s) = \frac{1}{2}\rho^T M^{-1} \rho + H_s(s) \tag{2}$$

where $\rho = M\dot{q}$ denotes the momenta of the internal degrees of freedom. It can be shown that (1) can then be expanded as:

$$\begin{bmatrix} \dot{s} \\ \dot{\rho} \\ \dot{q} \\ \tau_r \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & A(q,r) & 0 & B(q,r) \\ -A^T(q,r) & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -B^T(q,r) & 0 & 0 & 0 \end{bmatrix}}_{D(q,r)} \begin{bmatrix} \frac{\partial H}{\partial s} \\ \frac{\partial H}{\partial \rho} \\ \tau \\ \dot{r} \end{bmatrix} \tag{3}$$

Where $D(q,r)$ is a skew-symmetric matrix representing the extended Dirac structure. The power conjugate pair $(\dot{q}, \tau)$ defines the new control port.

2.2 Change of Coordinates

The behavior of a variable stiffness actuator, seen at the output, is essentially the behavior of a *linear* spring, of which the equilibrium position and the stiffness can be changed (Palli et al., 2008). We propose a change of coordinates

$$S : q \mapsto \tilde{q}, \quad \tilde{q} := (\bar{r}, k) \tag{4}$$

where $\bar{r}$ denotes the equilibrium output position, and k the apparent output stiffness, defined hereafter, to capture the behavior of the variable stiffness actuator in terms of these quantities.

The equilibrium position $\bar{r}$ of the actuator output is, by definition, the position r for which $\dot{r} = 0$ and remains zero, i.e., the position r for which the potential (elastic) energy function attains a minimum. Given a configuration q of the internal degrees of freedom, and $\dot{q} = 0$, we thus have

$$\begin{aligned} \bar{r} &= \arg\min_r H(\rho, q, r) \\ &= \arg\min_r (H_s \circ \lambda)(q, r) \end{aligned} \tag{5}$$

where the second equality follows from q and r being stationary and from the Hamiltonian energy function being quadratic in the momentum variables. The apparent output stiffness follows from the definition of the stiffness:

$$k := \frac{\delta \tau_r}{\delta r} \tag{6}$$

i.e., the infinitesimal force generated by an infinitesimal change in output position. Note that the stiffness is a local property, thus (6) is only valid for stationary configurations $\dot{q} = 0$ and $\dot{r} = 0$. From (3) and the kinematic relation λ, the force at the output is $\tau_r = \frac{\partial}{\partial r} H_s(s \circ \lambda)$, and thus we obtain that the apparent output stiffness is given by:

$$\begin{aligned} k &= -\frac{\partial^2 H}{\partial r^2}(q, r) \\ &= -\frac{\partial^2 (H_s \circ \lambda)}{\partial r^2}(q, r) \end{aligned} \tag{7}$$

where the second equality again follows from q and r being stationary. Observe that both $\bar{r}$ and k are not functions of r, as they are only defined for a particular value of r.

Using (5) and (7), the change of coordinates S is obtained, and it follows that, for a given configuration (q, r),

$$\dot{\tilde{q}} = \frac{\partial S}{\partial q}\dot{q} \tag{8}$$

Assumption 1. The change of coordinates (8) is a diffeomorphism and independent of q. •

Commonly encountered variable stiffness actuator designs, for example the antagonistic design that is the basis of the VSA-I presented by Tonietti et al. (2005), or the design of the VS-Joint by Wolf and Hirzinger (2008), satisfy this assumption. Under this assumption, the dynamics (3) can be rewritten in the new coordinates by transforming the momenta ρ into the new coordinates, denoted by $\tilde{\rho}$. Correspondingly, the new inertia matrix $\tilde{M}$ is obtained as

$$\tilde{M} = \left(\frac{\partial S}{\partial q}\right)^{-T} M \left(\frac{\partial S}{\partial q}\right)^{-1} \tag{9}$$

Then, in the new coordinates, the Hamiltonian energy function becomes

$$\tilde{H}(\tilde{\rho}, \tilde{q}, r) = \frac{1}{2}\tilde{\rho}^T \tilde{M}^{-1}\tilde{\rho} + (H_s \circ \lambda)(S^{-1}(\tilde{q}), r)$$

The dynamic equations in port-Hamiltonian form readily follow, with a new control port $(\tilde{\tau}, \dot{\tilde{q}})$.

For notational convenience, we introduce the following variables for the remainder of this paper:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} r \\ \dot{r} \\ \bar{r} \\ k \end{bmatrix}$$

Proposition 1. Due to the change of coordinates (4), the behavior of the variable stiffness actuator can be described in the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{m}x_4(x_3 - x_1) \\ \dot{x}_3 &= u_1 \\ \dot{x}_4 &= u_2 \end{aligned} \tag{10}$$

or:

$$\dot{x} = f(x) + g_1\, u_1 + g_2\, u_2$$

where $x \in \mathcal{M}$ denotes the state as element of the state manifold $\mathcal{M}$, $f(x)$ is the drift vector field, and g_1 and g_2 are the constant control input vector fields. •

Remark 1. We note that, since x_4 corresponds to a stiffness, which is a positive definite quantity, the state manifold $\mathcal{M}$ has a border. Therefore, any solution that is obtained in what follows is only valid if it remains in the bounded set $\{x \in \mathcal{M} \mid x_4 > 0\}$. ◁

3. PROBLEM FORMULATION

The goal of this work is to embed desired behavior into the variable stiffness actuator as much as possible. Consequently, the desired behavior should be represented in the form of a dynamical system, so that it is meaningful to let the autonomous part of (10) approach the desired behavior.

Assumption 2. The desired periodic motion of the actuator can be described in the phase space (x_1, x_2) by a dynamical system of the form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a(x_1) - \gamma \end{aligned} \tag{11}$$

implying that the desired motion $x_1(t)$ is bounded and at least twice continuously differentiable. •

Assumption 3. Because (10) describes a physical system, the desired motion is such that it can be achieved with finite inputs u. •

Remark 2. Note that we do not allow $a(\cdot)$ to be a function of x_2, as this would imply damping, which will not result in a periodic motion. Only if this damping is nonlinear and satisfies certain properties, such a description can result in periodic behavior. For example, the solutions to Liénard systems are, when certain conditions are met, limit cycles in the phase space (Strogatz, 1994). This topic is considered beyond the scope of this paper. ◁

By combining (10) and (11), we can define a function $\Gamma(x)$:

$$\Gamma(x) = \tfrac{1}{m}x_4(x_3 - x_1) - a(x_1) \tag{12}$$

i.e., $\Gamma(x)$ defines which output force must be generated to follow a desired trajectory. It follows that $\Gamma(x)$ defines foliations $\mathcal{N}_\gamma \subset \mathcal{M}$ described by

$$\mathcal{N}_\gamma = \{x \in \mathcal{M} \mid \Gamma(x) = \gamma\}$$

The goal of this work is to find an input u, such that the system remains on the foliation for a given desired motion (11), with minimum effort, i.e., with the smallest control input and minimal deviations of x_3 and x_4 from the initial conditions. This can be formally stated as follows.

Problem 1. *Given a desired motion of the actuator output, described by* (11)*, find initial conditions* $x^\circ = x(0)$*,* $x^\circ \in \mathcal{N}_\gamma$*, and a control input* u*, such that the criterion*

$$J = \int_0^\infty \frac{1}{2}\|x_3^\circ - x_3\|_{\bar{r}}^2 + \frac{1}{2}\|x_4^\circ - x_4\|_k^2 + \frac{1}{2}\|u\|_u^2\, dt \tag{13}$$

is minimized for given weighted 2*-norms* $\|\cdot\|_*$*.*

The integrand of (13) can be interpreted as a Hamiltonian energy function. Thus, minimizing J corresponds to finding the optimal trajectories $x_3(t)$ and $x_4(t)$ with respect to this energy function. Note that there are two parts in the integrand, and optimizing J means finding a trade-off between those two parts: the first two terms associate a cost to the deviation from the initial conditions, and the last term associates a cost to the rate of change of x_3 and x_4. In particular, the first two terms in fact formulate a measure of embodiment of the desired behavior into the variable stiffness actuator: if the deviations from the initial conditions of x_3 and x_4 are small, it means that the initial values for x_3 and x_4 result in an intrinsic passive behavior that is already close to the desired behavior.

Remark 3. It will be shown that there is a cost associated to u that naturally arises from physical considerations. However, there is no such physical cost for the deviations of x_3 and x_4. These costs are associated with changing the equilibrium position and the stiffness, and may be defined by the design of the actuator, by an analysis of desired disturbance rejection, or some other analysis. ◁

4. NOMINAL SOLUTION

The first step in solving Problem 1 is showing that there exists at least one solution to the problem. The approach will be in two parts: first we establish that there exists an input u such that the system (10) exhibits the desired motion (11) in the plane (x_1, x_2), and then we will infer that the obtained solution curves allow a minimization of the criterion (13).

4.1 Nominal Control Input

Given the desired motion (11) and the corresponding $\Gamma(x)$ as defined in (12), we extend the system description (10) by defining an output function $h(x) = \Gamma(x) - \gamma$. Then, given initial conditions $x^\circ \in h^{-1}(0)$, it is possible to compute the maximal controlled invariant output-nulling submanifold. In particular, following the algorithm presented by Nijmeijer and van der Schaft (1990), we obtain the following.

First, we define the submanifold $\mathcal{Z}_1 \subset \mathcal{N}_\gamma \subset \mathcal{M}$ by

$$\mathcal{Z}_1 = \{x \in \mathcal{M} \mid h(x) = 0\}$$

This submanifold is of dimension three, because the restriction of the output function being zero defines a curve of dimension one. With $x^\circ \in \mathcal{Z}_1$, the system dynamics remain in $\mathcal{Z}_1$ for all time, if $\dot{h}(x) = 0$ for all time. We calculate

$$\begin{aligned} \frac{d}{dt}h(x) &= L_f h(x) + L_{g_1} h(x)\, u_1 + L_{g_2} h(x)\, u_2 \\ &= L_f\Gamma(x) + [L_{g_1}\Gamma(x) \ , \ L_{g_2}\Gamma(x)] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

with

$$L_f\Gamma(x) = \left(-\frac{1}{m}x_4 - \frac{\partial a}{\partial x_1}\right) x_2$$

i.e., the Lie-derivative of $\Gamma(x)$ along the drift vector field $f(x)$, and similarly the Lie-derivatives of $\Gamma(x)$ along the control input vector fields:

$$[L_{g_1}\Gamma(x) \ , \ L_{g_2}\Gamma(x)] = \left[\tfrac{1}{m}x_4 \ , \ \tfrac{1}{m}(x_3 - x_1)\right] =: \mathcal{A}(x)$$

Since, as remarked before, x_4 is always nonzero, $\mathcal{A}(x)$ nonsingular for all x, and thus there always exists an input u, such that $\mathcal{A}(x)u + L_f\Gamma(x) = 0$. Define the submanifold $\mathcal{Z}_2 \subset \mathcal{Z}_1$ by

$$\mathcal{Z}_2 = \{x \in \mathcal{Z}_1 \mid \dot{h}(x) = 0\}$$

This submanifold is of dimension two, due to the added restriction of $\dot{h}(x) = 0$. We can take for u a combination of a state feedback and a new input v:

$$u = -\mathcal{A}^R(x)\, L_f\Gamma(x) + \mathcal{A}^\perp(x)v \qquad (14)$$

where $\mathcal{A}^R$ denotes a right inverse of $\mathcal{A}$, and $\mathcal{A}^\perp$ is the annihilator for $\mathcal{A}$. Then, for initial conditions $x^\circ \in \mathcal{Z}_2$, the input (14) ensures that the system remains in $\mathcal{Z}_2$, and thus $\mathcal{Z}_2 =: \mathcal{Z}^*$ is the maximal controlled invariant output-nulling submanifold. Or, in other words, any trajectory $x(t)$ that is a solution to (10), subject to the input (14), will show the desired motion (11) in the plane (x_1, x_2) if the initial conditions $x^\circ \in \mathcal{Z}^*$.

Remark 4. Because $\mathcal{A}(x)$ is not full rank, (14) defines infinitely many solutions. It is well known that the weighted pseudo inverse is a right inverse that gives a solution of minimum norm with respect to a metric (Ben-Israel and Greville, 2003). Therefore, in the context of optimizing (13), in (14) the pseudo inverse should be taken with respect to the metric defining the norm $\|\cdot\|_u$, and $v \equiv 0$.

Inspection of the third row of (3) reveals that, at the port $(\dot{q}, \tau)$, the infinitesimal change $\delta\dot{q}$ as a result of an infinitesimal change of applied control torque $\delta\tau$ is, using the energy function (2), given by:

$$\frac{\delta\dot{q}}{\delta\tau} = \frac{\partial^2 H}{\partial\rho^2} = M^{-1}$$

Hence, the metric defined by M is a useful metric to measure a change of $\dot{q}$. Since the input $u = \dot{\tilde{q}}$ is defined in the new coordinates $\tilde{q}$, a meaningful choice for the metric inducing the norm $\|\cdot\|_u$ is the pseudo mass matrix $\tilde{M}$ defined in (9). Using the definition of the norm, we obtain

$$\frac{1}{2}\|u\|^2_{\tilde{M}} = \frac{1}{2}u^T\tilde{M}u$$

which has indeed the units of energy, as we observed in defining the integrand of (13). ◁

4.2 Bounded Solutions

Optimization of the criterion (13) is only meaningful if the solutions $x_3(t)$ and $x_4(t)$ remain in some bounded neighborhood of the initial conditions (x_3°, x_4°). It is tempting to assume, since the motion in the plane (x_1, x_2) is periodic, that also the solutions (x_3, x_4) are periodic. However, this assumption may not be valid, since the solution $x(t)$ can be chaotic while still the projection onto (x_1, x_2) gives the desired periodic motion. However, by investigation of (14), we can deduce some properties of the solution $x(t)$ of the system (10), (14).

The desired motion is described in (11) by a dynamic system without damping, and thus of the form $\ddot{x}_1 = a(x_1) - \gamma$. For $v \equiv 0$, (14) is differentiable, because $\mathcal{A}(x)$ is nonsingular (see Remark 1) and bounded by Assumption 3. Defining $z := (x_1, x_3, x_4)$, it follows that on a closed and finite time interval, $\dot{z}$ is bounded and C^1, and that $\ddot{z}$ is finite. Therefore, we can write the system dynamics as

$$\ddot{z} = F(z)$$

where $F(z)$ is a function that depends on the state z of the system only, because due to (11) and the restriction to $\mathcal{Z}^*$, the feedback in (14) can be found in terms of x_1, x_3, x_4 only. If it is possible to find a potential energy function $U(z)$ such that

$$F(z) = -\frac{\partial U}{\partial z}$$

then the system is conservative. Then, assuming $\dot{z}(0) = 0$, due to the law of conservation of energy all solutions $z(t)$ remain in the ellipse $U(z) \le U(z(0))$ (Arnol'd, 1989).

However, $U(z)$ may not exist, or it may be unbounded or not smooth. Therefore, at this point, no strict conclusions may be drawn about the boundedness of the solutions of $x(t)$. But, if $U(z)$ does not exist and thus that the system is not conservative, this implies that there is energy injection or dissipation, which may be countered by a proper choice for the additional input v. As stated, the optimization of (13) is only meaningful if the solution $x(t)$ is bounded, and therefore we will assume the following.

Assumption 4. For a desired motion described by (11), starting from initial conditions $x^\circ \in \mathcal{Z}^*$, there is a v such that the solution $x(t)$ of the system (10), (14), remains within an ellipse defined by the initial conditions. •

Remark 5. The preceding analysis implies that $v \equiv 0$ should give the desired behavior. Simulation examples in Section 6 show that this is indeed the case for some nontrivial periodic motions in (x_1, x_2). ◁

5. OPTIMIZATION

In the previous Section, it was established that the maximal controlled invariant output-nulling submanifold $\mathcal{Z}^*$ is of dimension 2. One dimension corresponds to the desired motion in (x_1, x_2), leaving one degree of freedom in (x_3, x_4). As mentioned before, taking the pseudo inverse in (14) with respect to the desired norm on u in (13) results in that the term $\frac{1}{2}\|u\|_u^2$ is already minimized among all possible solutions. This leaves the remaining two terms to be minimized by a proper choice of the initial conditions (x_3°, x_4°).

In general, the optimization problem cannot be solved analytically. Therefore, we propose a variation on the line search algorithm that exploits the fact that there is only one degree of freedom left in the choice for the initial conditions (x_3°, x_4°), due to the restriction to $\mathcal{Z}^*$. It is not possible to find the gradient descent of J with respect to (x_3°, x_4°) analytically, but since there is only one degree of freedom to search in, the following algorithm can be effectively executed.

Step 0: Choose initial conditions (x_1°, x_2°) for the duration of the algorithm. Determine an initial guess for $x_3^{\circ,1}$ and calculate $x_4^{\circ,1}$ such that $x^{\circ,1} \in \mathcal{Z}^*$. Choose an initial step-size ε_1.

Step $k > 0$: Determine the gradient descent direction:

- Calculate the cost J_k according to (13), by simulating the system (10), (14) for a number of periods, starting from initial conditions $x^{\circ,k}$.
- Compute J_k^+ and J_k^- by computing the costs starting from initial conditions $x_3^{\circ,k} + \varepsilon_k$ and $x_3^{\circ,k} - \varepsilon_k$ (and corresponding initial values x_4) respectively.
- Determine $J_{k+1} = \min(J_k^+, J_k^-)$.
- If $J_k - J_{k+1}$ is smaller than a threshold δ, terminate the algorithm: the corresponding initial conditions minimize (13). Otherwise, continue with the corresponding initial values and set $\varepsilon_{k+1} = \varepsilon_k/2$.

Remark 6. Successful convergence of this algorithm relies on the assumption that, since J is quadratic, it has at least a local minimum. Reaching the global minimum depends on the initial guess $x_3^{\circ,1}$, the threshold value δ, and the initial stepsize ε_1. ◁

6. EXAMPLES

In this Section, we illustrate the results of the preceding Sections with two examples. The first example is the harmonic oscillator, which has optimal values for (x_3°, x_4°) that can be determined a priori. In the second example, the Duffing oscillator is used to illustrate the effectiveness of the optimization process for nontrivial periodic motions.

6.1 The Harmonic Oscillator

The harmonic oscillator with unit frequency and amplitude is described in the form (11) as:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1$$

which admits the analytical solution

$$x_1(t) = \cos t, \quad x_2(t) = -\sin t$$

It follows that the submanifold $\mathcal{Z}_1$ is defined as

$$\mathcal{Z}_1 = \{x \in \mathcal{M} \mid \tfrac{1}{m} x_4(x_3 - x_1) + x_1 = 0\}$$

Table 1. Parameter values

Desired motion	(x_1°, x_2°)	$(-2, 0)$
	α, β, γ	$1, -2, 3$
Cost criterion	$\|\cdot\|_{\bar{r}}$	$\|\cdot\|$ (2-norm)
	$\|\cdot\|_k$	$\|\cdot\|$ (2-norm)
	$\|\cdot\|_u$	$\|\cdot\|_{\tilde{M}}$ (weighted 2-norm with $\tilde{M} = \text{diag}(1, 4)$)
Optimization	δ	0.005
	ε_1	0.2

It is well known that the harmonic oscillator can be implemented by tuning the stiffness to the frequency of the desired oscillation, i.e., choosing k such that $\omega = \sqrt{k/m}$. In this example, we have $m = 1$ and $\omega = 1$, and thus taking $k = 1$ gives the desired motion. Indeed, taking $x_3^\circ = 0$ and $x_4^\circ = 1$ results in initial conditions that are in $\mathcal{Z}_1$ for any (x_1°, x_2°). Moreover, we compute $L_f\Gamma(x)$, with $a(x_1) = -x_1$, we obtain

$$L_f\Gamma(x) = \left(-x_4 - \frac{\partial a}{\partial x_1}\right) x_2 = (-x_4 + 1)x_2$$

which is always equal to zero for $x_4 = 1$. Hence, the control input u can remain zero, and the cost (13) is trivially minimized. This illustrates perfectly the principle of embedding desired behavior.

6.2 The Duffing Oscillator

Duffing's equation was originally introduced to model nonlinear oscillations with a hardening stiffness effect, but it provides in general an example for studying nonlinear oscillations (Guckenheimer and Holmes, 1983). In this example, the undamped Duffing oscillator is considered, which takes the form:

$$\ddot{x}_1 + \beta x_1 + \alpha x_1^3 = \gamma \tag{15}$$

with $\alpha > 0$. Oscillations of this type can be formulated in the form of (11) by taking $a(x_1) = -\beta x_1 - \alpha x_1^3$. To compute the maximal controlled invariant submanifold $\mathcal{Z}^* \subset \mathcal{N}_\gamma \subset \mathcal{M}$, we first compute

$$\mathcal{Z}_1 = \{x \in \mathcal{M} \mid \Gamma(x) + \gamma = 0\}$$

with

$$\Gamma(x) = -\beta x_1 - \alpha x_1^3 - x_4(x_3 - x_1)$$

It follows, for any initial conditions (x_1°, x_2°), that (x_3°, x_4°) must satisfy:

$$x_4^\circ = \frac{-\beta x_1^\circ - \alpha x_1^{\circ 3} + \gamma}{x_3^\circ - x_1^\circ}$$

Using this relation, the algorithm presented in Section 5 is executed, with the parameter values presented in Table 1. Setting the cost for both x_3 and x_4 equally in the cost criterion (13) implies that there is no preference on using either one. Note, however, that there are different costs associated with dynamically changing x_3 and x_4, indicated by $\tilde{M} = \text{diag}(1, 4)$. For the initial guess of $x_3^{\circ,1}$, we take the average of the solution to (15).

Figure 1 shows the solution curves of the system (10), (14) in the plane (x_1, x_2). Both the initial solutions and the optimal solution, according to the algorithm of Section 5, are shown, but cannot be distinguished. The $\circ$ indicates the initial guess $x_3^{\circ,1}$ for the algorithm, and the $+$ the optimal x_3°. Figure 2 shows the components x_3 and x_4

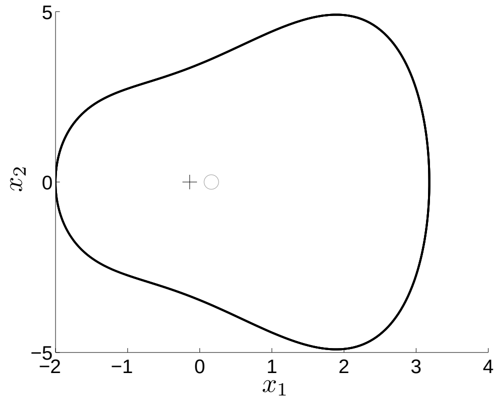

Fig. 1. Solutions in the plane (x_1, x_2) - The $\circ$ indicates the initial guess $x_3^{\circ,1}$, and the $+$ the optimal initial value for x_3.

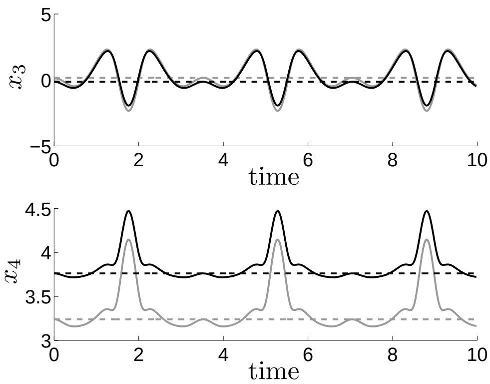

Fig. 2. Solutions (x_3, x_4) - The grey curves correspond to the first step of the algorithm of Section 5, the black curves to the optimal solution, with the dashed lines indicating the initial values.

of both solutions. It can be clearly seen that the optimal solution (solid black curve) achieves smaller excursions from the initial conditions (dashed curves). It is noted that both the solutions x_3 and x_4 are reduced comparatively (observe the scales of the vertical axes).

The cost criterion (13) is calculated over a time span of 100 s. For the optimal solution found by the algorithm, a numerical value of $J = 141.374$ is found. A fine-gridded brute force calculation of the cost for all possible initial conditions finds a minimum of $J = 141.316$, illustrating the effectiveness of the algorithm. To illustrate that it makes sense to have a varying stiffness, the process is repeated with the same parameter values, but with $u_2 \equiv 0$, i.e. a fixed stiffness. The algorithm then finds an optimal cost of $J = 158.397$ (brute force: $J = 155.998$), which is higher than obtained with the variable stiffness, even though we assigned a higher cost to the dynamic changes of the stiffness with respect to the equilibrium position.

7. CONCLUSIONS AND FUTURE WORK

In this paper, a cost criterion was proposed, that formulates a measure of embodiment of desired behavior into a variable stiffness actuator. In particular, minimization of the cost criterion achieves a desired output motion with minimum control effort. The effectiveness of this approach was illustrated in an algebraic and a simulation example. Currently, the algorithm is being implemented on a test setup, and experimental results will be reported in a future article.

Future work will focus on how the behavior of the variable stiffness actuator should change in case of a disturbance, considering the fact that the disturbance will add energy to the system that may be used efficiently for actuation. Furthermore, extensions to multi degree of freedom systems need to be formulated. Rather than considering each degree of freedom separately, it should be investigated if a coordinated approach to embodiment of desired behavior in all degrees of freedom is possible.

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