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Optimal Feedback Control for Route Tracking with a Bounded–Curvature Vehicle *

Philippe Souères
LAAS - CNRS
7 Av. du colonel Roche
31077 Toulouse Cedex 4
FRANCE
soueres@laas.fr

Andrea Balluchi
PARADES E.E.I.G
Via San Pantaleo, 66
00186 Roma
ITALY
balluchi@parades.rm.cnr.it

Antonio Bicchi
Centro Interdipartimentale
di Ricerca "Enrico Piaggio"
Univ. di Pisa
56100 Pisa, ITALY
bicchi@ing.unipi.it

Abstract

We consider the kinematic model of a vehicle moving forward with a lower bounded turning radius. This model, is relevant to describe the kinematics of road vehicles as well as aircraft cruising at constant altitude, or sea vessels. We consider the problem of minimizing the length travelled by the vehicle starting from a generic configuration to connect to a specified route. A feedback law is proposed, such that straight routes can be approached optimally, while system is asymptotically stabilized. Experimental results are reported showing real–time feasibility of the approach.

1 Introduction

We consider the problem of driving a wheeled robot with a constraint on the turning angle along a given route. The model ignores the vehicle dynamics, however, it explicitly takes into account inherent kinematic limitations of automobiles along highways and aircraft cruising at constant altitude. It was shown that the kinematic model of a vehicle that can drive both forward and backward with bounded curvature is locally controllable. A vehicle that can only move forward and is subject to curvature bounds is still controllable, although not small time locally controllable. For this latter type of vehicle, Dubins [6] studied the shortest paths joining two arbitrary configurations. He proved that optimal paths are made with at most three pieces of either type "C" (arcs of circle with minimal radius R), or type "S" (straight line segments). Furthermore, Dubins showed that optimal paths necessarily belong to the following sufficient family:

$$\{C_a C_b C_e \ , \ C_u S_d C_v\} \tag{1}$$

where the subscripts, indicating the length of each piece, are respectively restricted to:

$$b \in (\pi R, 2\pi R); \quad a, e \in [0, b], \quad u, v \in [0, 2\pi R), \quad d \geq 0 \tag{2}$$

Later, [12], and [2], gave a new proof of Dubins' result using Optimal Control Theory. Finally, on the basis of these works, the complete synthesis of optimal paths was constructed by Bui *et al* [3]. Here, we consider the problem of tracking a reference route for the kinematic model of Dubins. Trajectory or route tracking control has been widely studied for nonholonomic systems[1] [10], [4], [7], [5]. However, as opposite to work in route planning, most of the vehicle tracking control literature did not consider curvature bounds in the vehicle model. In our work, we consider the design of a route tracking control law for Dubins' vehicle, for which the forward velocity profile is given. The work is based on the characterization of optimal paths to reach tangentially a rectilinear route. We propose a feedback control that locally stabilizes the vehicle on the route, and guarantees global convergence in finite–time. The optimal control result is based on Pontryagin's Maximum Principle (PMP) and we recur to tools of hybrid control theory to prove Lyapunov stability of the feedback control law.

In §2 we state the problem and determine a sufficient family of optimal paths. In §3 we synthesize a feedback controller that implements optimal curves, formalize it as an hybrid control system, and prove its Lyapunov stability. Experimental results are presented in §4.

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[1] By "route" or "path" we refer to a curve in the plane where the vehicle moves. By "trajectory" we mean a route with an associated time law.

2 Shortest paths to join straight routes

Let the configuration of the vehicle be described as $X = (M, \theta) \in \mathbb{R}^2 \times S^1$, where $M = (x, y)$ are the coordinates of the reference point of the vehicle with respect to a reference frame, θ is the heading angle with respect to the frame x-axis. The kinematics of the vehicle is described by

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \omega \end{cases} \tag{3}$$

where v and ω are the linear and angular velocities of the vehicle. Without loss of generality, up to a time-axis rescaling (cf. e.g. [9]), we assume that $\dot{v}(t) = 0$, $v(t) \equiv V$. The turning radius of the vehicle is lower bounded by a constant value $R > 0$, which results in an upper bound on the vehicle's angular velocity ω as

$$|\omega| < \frac{V}{R}. \tag{4}$$

Let $\mathcal{T}$ be a target rectilinear path in the plane, with a prescribed direction of motion determined by the angle $\alpha \in [-\pi, \pi]$ with respect to the x-axis. We consider the optimal control problem:

$$\text{Minimize } J = \int_0^T \sqrt{\dot{x}^2 + \dot{y}^2}dt = VT, \tag{5}$$

subject to (3) and (4), with $X(0) = (M_0, \theta_0)$ and such that, at the unspecified terminal time T, $M(T) \in \mathcal{T}$ and $\theta(T) = \alpha$.

2.1 Characterization of optimal arcs

As v is constant, the problem is equivalent to a minimum time problem. We first characterize extremal solutions by applying (PMP) [8] along the lines of [12], [2], and [11], then refine the results with a global geometric analysis. The Hamiltonian function associated to our problem is

$$\mathcal{H}(\psi, X, \omega) = \psi_1 V\cos\theta + \psi_2 V\sin\theta + \psi_3\omega \tag{6}$$

where $\psi = (\psi_1, \psi_2, \psi_3)^T$ is the system costate. According to PMP, a necessary condition for the control $\omega^*(t)$ to be optimal is that there exists a negative constant ψ_0, such that at all times $t \in [0, T]$

$$-\psi_0 = \mathcal{H}(\psi(t), X(t), \omega^*(t)) = \max_{\omega\in\Omega} \mathcal{H}(\psi(t), X(t), \omega(t)) \tag{7}$$

where Ω is the class of piecewise continuous controls on $[0, T]$ taking values in $[-\frac{V}{R}, +\frac{V}{R}]$. Writing the adjoint equations: $\dot{\psi}(t) = -\frac{\partial\mathcal{H}}{\partial X}(\psi(t), X(t), \omega^*(t))$ we deduce:
- ψ_1 and ψ_2 are constant
- $\psi_3(t) = \psi_3(0) + \psi_1(y(t) - y(0)) - \psi_2(x(t) - x(0))$,

ψ_3 is the switching function of ω^*. As described in [11], two cases may occur along an optimal path:

1. ψ_3 only vanishes at isolated points of $[0, T]$. In that case, $|\omega| = \frac{V}{R}$ and the sign of ω is opposite to the sign of ψ_3. The path is made of arcs C.

2. ψ_3 vanishes over a nonzero interval $I \subset [0, T]$:

$$-\psi_0 = \psi_1 V\cos\theta + \psi_2 V\sin\theta, \quad \forall t \in I. \tag{8}$$

   Hence, $\theta(t)$ is constant along I, the motion is a line segment S.

Control switches between two C (points of inflection) or between a C and a S, occur when ψ_3 vanishes. $\psi_3(t) = 0$ determines a plane in $\mathbb{R}^2 \times [-2\pi, 2\pi]$:

$$\psi_1 y(t) - \psi_2 x(t) + \psi_3(0) - \psi_1 y(0) + \psi_2 x(0) = 0 \tag{9}$$

Equation (9) defines a line $\mathcal{D}$ on the plane of the vehicle's motion, a tangent vector of which is given by $(\psi_1, \psi_2)^T$. $\mathcal{D}$ is supporting the line segments and the points of inflection. In our optimal control problem, the transversality condition states that, at unspecified final time T, $\psi(T) \perp \mathcal{T}$, then we deduce the relation:

$$\cos(\alpha)\psi_1(T) + \sin(\alpha)\psi_2(T) = 0. \tag{10}$$

Hence, we have the following:

Lemma 1 *Optimal paths solving problem (5) belong to the Dubins' family (1), and are such that rectilinear segments and points of inflection belong to a same line $\mathcal{D}$ perpendicular to the target line $\mathcal{T}$.*

On the basis of lemma 1, the family (1) and can be further refined using geometric arguments.

2.2 Refinement of the sufficient family

To specify the vehicle's direction of motion, we replace the (C) by (l) for left turn or by (r) for right turn. Each path will be represented by a word belonging to the family $\{lrl, rlr, lsl, rsr, rsl, lsr\}$. Subscripts are used to specify the length of each piece.

Property 1 *In the plane of the vehicle's motion, let Δ be the line of equation $y\cos\frac{\theta}{2} + x\sin\frac{\theta}{2} = 0$, and M,*

M' *any two points symmetric with respect to* Δ. *If* γ *is a path starting from* $(O, 0)$ *and ending at* (M, θ), *there exists a path* γ' *isometric to* γ, *starting from* $(O, 0)$ *and ending at* (M', θ). *The word of* γ' *is obtained by writing the word of* γ *in reverse direction.*

Remark 1 *Saying that the paths* γ *and* γ' *are isometric means that they can be deduced from each other by an isometric transformation. Such paths have obviously the same length.*

Property 1 is illustrated by figure 1. The proof[2] can be directly deduced from [11], lemma 1, p 676.

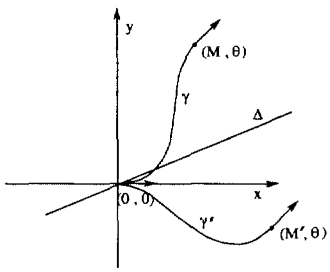

Figure 1: Symmetry in plane of the vehicle's motion

When the final orientation of paths is π, the line Δ is identical to the y-axis and the directed points (M, π) and (M', π) belong necessarily to the same directed line $\mathcal{T}$, parallel to the x-axis (see figure 2). From property 1 and remark 1 both paths are equivalent for linking $(O, 0)$ to $\mathcal{T}$. This result can be stated in a more general way as follows:

Property 2 *If* γ *is a path starting from* (M, θ) *and reaching* $\mathcal{T}$ *with orientation* $\alpha = \theta \pm \pi$, *there exists an isometric path* γ' *from* (M, θ) *to* $\mathcal{T}$. *The word of* γ' *is obtained by writing the word of* γ *in the reverse direction.*

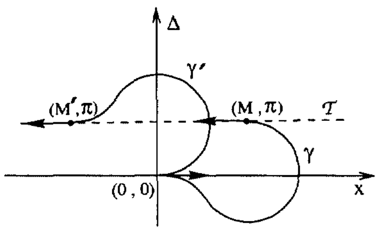

Figure 2: Isometric paths $r_{(\pi R+a)}l_a$ and $l_a r_{(\pi R+a)}$

[2]The problem was stated in terms of ending at $(O, 0)$, but the reasoning for our problem is similar.

On the base of lemma 1, properties 1 and 2, further properties can be deduced. In each case the proof follows the same reasonning. To prove that a path is not optimal we show that it is equivalent to a nonoptimal path obtained by replacing a part of the initial path with an isometrical one.

Property 3 *A path CCC is never optimal for reaching tangentially a directing line.*

Proof: Consider a path $l_a r_b l_e$. Replacing the final part along which the variation of θ equals π by an isometric one according to property 2, we get an equivalent path $l_a r_e l_e r_{(\pi R+e)}$ not optimal as the point of inflection does not belong to $\mathcal{D} \perp \mathcal{T}$. □

According to (1) and property 3 it suffices to consider paths CSC (and subpaths CC, C and SC). A path S_d with $d > 0$ is trivially not optimal.

Property 4 *A necessary condition for a path* C_a *to be optimal is that* $a \leq \frac{3\pi}{2}R$

Proof: If $a = \frac{3\pi}{2}R + \varepsilon$ there exist an equivalent path $C_\varepsilon C_{\pi R} C_{\frac{\pi}{2}R}$ which is not optimal because the line $\mathcal{D}$ containing the two points of inflection is parallel to the target line. □

Property 5 *A necessary condition for a path* $C_a C_b$ *to be optimal is that one of the following two conditions be verified:*

- $b \in [0, \frac{\pi}{2}R]$ *and* $a \in [0, b + \pi R]$,
- $b \in [\pi R, \frac{3\pi}{2}R]$ *and* $a \in [0, b - \pi R]$,

Proof: Consider a path $l_a r_b$. If $\frac{\pi}{2}R < b < \pi R$, the line $\mathcal{D} \perp \mathcal{T}$ cuts the last arc r at a point where the control ω does not switch and contradicts the necessary condition. Then $b \in [0, \frac{\pi}{2}R]$ or $b \in [\pi R, \frac{3\pi}{2}R]$. If $b \in [0, \frac{\pi}{2}R]$ and $a = b + \pi R + \varepsilon$, $\varepsilon > 0$, one can replace the final part $l_{(b+\pi R)} r_b$ between $(M, \alpha - \pi)$ and $\mathcal{T}$ by a equivalent path $r_b l_{(b+\pi R)}$. The path $l_a r_b$ is then equivalent to a path $l_\varepsilon r_b l_{(b+\pi R)}$ which is not optimal according to property 3. Then, necessarily $a \in [0, b + \pi R]$. If $b \in [\pi R, \frac{3\pi}{2}R]$ and $a = b - \pi R + \varepsilon$, one can replace the last part between $(M, \alpha - \pi)$ and $\mathcal{T}$ by an equivalent path $r_b l_{a-\varepsilon}$. The path $l_a r_b$ is equivalent to a path $l_\varepsilon r_b l_{a-\varepsilon}$ which is not optimal according to property 3. Then, necessarily $a \in [0, b - \pi R]$. □

Property 6 *A necessary condition for a path* $C_a S_d C_b$ *to be optimal is that the segment* S *be perpendicular to* $\mathcal{T}$, $a \in [0, \pi R]$ *and* $b = \frac{\pi}{2}R$.

C_a	r_a l_a	$a \in (0, \frac{3\pi}{2} R]$
$C_a C_b$	$l_a r_b$ $r_a l_b$	$b \in (0, \frac{\pi}{2} R]$ and $a \in (0, b + \pi R]$ or $b \in [\pi R, \frac{3\pi}{2} R]$ and $a \in (0, b - \pi R]$
$C_a S_d C_{\frac{\pi}{2}R}$	$r_a s_d r_{\frac{\pi}{2}R}$ $l_a s_d r_{\frac{\pi}{2}R}$ $r_a s_d l_{\frac{\pi}{2}R}$ $l_a s_d l_{\frac{\pi}{2}R}$	$d > 0$ and $a \in [0, \pi R]$

Table 1: Sufficient family of extremal trajectories.

Proof: From lemma 1 the segment S belongs to $\mathcal{D} \perp \mathcal{T}$. Necessarily, $b = \frac{\pi R}{2}$ or $b = \frac{3\pi R}{2}$. If $b = \frac{3\pi}{2} R$, the path is equivalent to a nonoptimal path $CSC_{\pi R} C_{\frac{\pi}{2} R}$. Now, if $a = \pi R + \varepsilon$, $\varepsilon > 0$, we construct an equivalent path $C_\varepsilon C_{\pi R} S C_{\frac{\pi}{2} R}$ which is not optimal as it contains a point of inflection that does not belong to $\mathcal{D}$. □

At this stage, gathering the preceding properties we can reduce our search for optimal paths to the sufficient family described in table 1.

3 Optimal feedback control

To construct the optimal synthesis it is expedient to use path–based coordinates $s, \tilde{y}, \tilde{\theta}$: s is the curvilinear abscissa measuring the motion of the perpendicular projection of the robot reference point on $\mathcal{T}$. $\tilde{y}$ represents the distance between $\mathcal{T}$ and the robot reference point, divided by R. $\tilde{\theta} = \theta - \alpha$ is the heading angle error. The tracking problem (5) is reformulated in these variables as the minimum-time convergence of $(\tilde{y}, \tilde{\theta})$ to the origin $(0, 0)$ of the *reduced state space*. $\tilde{y}$ and $\tilde{\theta}$ obey the following dynamic:

$$\begin{aligned} \dot{\tilde{y}} &= \sin(\tilde{\theta}) \tfrac{V}{R} \\ \dot{\tilde{\theta}} &= \omega \end{aligned} \tag{11}$$

3.1 Optimal synthesis

For each path type contained in the sufficient family we compute, in the reduced space, the domains of possible initial points for paths ending at the origin$(0, 0)$. This construction reveals that the mapped domains are adjacents to each other i.e. they just share part of their boundaries. Furthermore, when two different paths start from the boundary of two adjacent domains, they both have the same length. Hence, an optimal synthesis in the $(\tilde{y}, \tilde{\theta})$ space is derived from table 1 choosing arbitrarily, when more than one solution exists, a particular one to obtain at the end a region $\mathcal{G}_{(\tilde{y}, \tilde{\theta})}$ which, modulo 2π on $\tilde{\theta}$, covers the whole reduced state space. The different cells making the partition of $\mathcal{G}_{(\tilde{y}, \tilde{\theta})}$ are defined by:

$$\begin{array}{ll}
\mathbf{r} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in (0, \frac{3}{2}\pi), \sigma_R(\tilde{y}, \tilde{\theta}) = 0\}, \\
\mathbf{l} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in (-\frac{3}{2}\pi, 0), \sigma_L(\tilde{y}, \tilde{\theta}) = 0\}, \\
\mathbf{rl}^{(1)} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in [0, \pi), \sigma_R(\tilde{y}, \tilde{\theta}) > 0, \sigma_P(\tilde{y}, \tilde{\theta}) \le 0\} \cup \{(0, \pi)\} \\
 & \cup \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in (-\frac{\pi}{2}, 0), \sigma_L(\tilde{y}, \tilde{\theta}) > 0, \sigma_P(\tilde{y}, \tilde{\theta}) \le 0\} \\
\mathbf{rl}^{(2)} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in (-\frac{3}{2}\pi, -\pi], \sigma_P(\tilde{y}, \tilde{\theta}) > 0, \sigma_L(\tilde{y}, \tilde{\theta}) < 0\} \\
\mathbf{lr}^{(1)} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in (0, \frac{\pi}{2}), \sigma_N(\tilde{y}, \tilde{\theta}) \ge 0, \sigma_R(\tilde{y}, \tilde{\theta}) < 0\} \\
 & \cup \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in (-\pi, 0], \sigma_N(\tilde{y}, \tilde{\theta}) \ge 0, \sigma_L(\tilde{y}, \tilde{\theta}) < 0\} \\
\mathbf{lr}^{(2)} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in [\pi, \frac{3}{2}\pi), \sigma_R(\tilde{y}, \tilde{\theta}) > 0, \sigma_N(\tilde{y}, \tilde{\theta}) < 0\} \\
\mathbf{rsr} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in (\frac{\pi}{2}, \frac{3}{2}\pi), \sigma_R(\tilde{y}, \tilde{\theta}) < 0\} \\
\mathbf{lsr} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2}), \sigma_N(\tilde{y}, \tilde{\theta}) < 0\} \\
\mathbf{rsl} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in (-\frac{\pi}{2}, \frac{\pi}{2}], \sigma_P(\tilde{y}, \tilde{\theta}) > 0\} \\
\mathbf{lsl} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{\theta} \in (-\frac{3}{2}\pi, -\frac{\pi}{2}), \sigma_L(\tilde{y}, \tilde{\theta}) > 0\} \\
\mathbf{sr} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{y} < -1, \tilde{\theta} = \frac{\pi}{2}\} \\
\mathbf{sl} & = \{(\tilde{y}, \tilde{\theta}) | \tilde{y} > +1, \tilde{\theta} = -\frac{\pi}{2}\},
\end{array} \tag{12}$$

where

$$\begin{aligned}
\sigma_R(\tilde{y}, \tilde{\theta}) &= \tilde{y} + 1 - \cos(\tilde{\theta}), & (13) \\
\sigma_L(\tilde{y}, \tilde{\theta}) &= \tilde{y} - 1 + \cos(\tilde{\theta}), & (14) \\
\sigma_N(\tilde{y}, \tilde{\theta}) &= \tilde{y} + 1 + \cos(\tilde{\theta}), & (15) \\
\sigma_P(\tilde{y}, \tilde{\theta}) &= \tilde{y} - 1 - \cos(\tilde{\theta}), & (16)
\end{aligned}$$

Proposition 1 *The twelve subsets* **l**, **r**, $\mathbf{rl}^{(1)}$, $\mathbf{lr}^{(1)}$, $\mathbf{rl}^{(2)}$, $\mathbf{rl}^{(2)}$, **rsr**, **lsr**, **rsl**, **lsl**, **sr**, **sl**, *plus* $\tilde{\mathrm{o}} = \{(0, 0)\}$ *determine a partition of the reduced state space, (mod. 2π on $\tilde{\theta}$) defining a synthesis of optimal paths.*

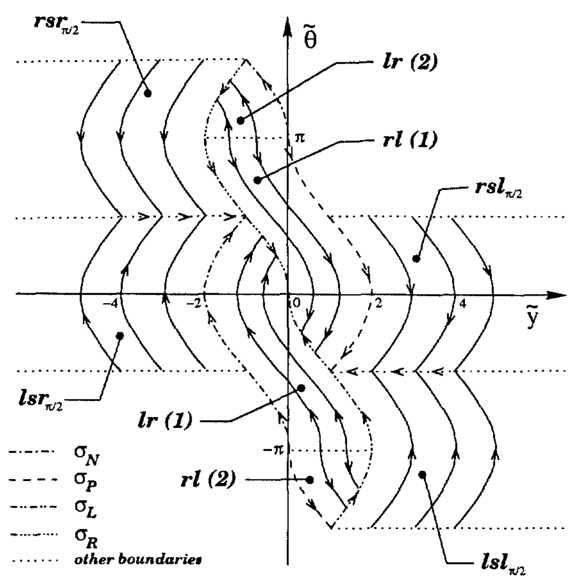

Figure 3: Optimal synthesis in the $(\tilde{y}, \tilde{\theta})$-plane.

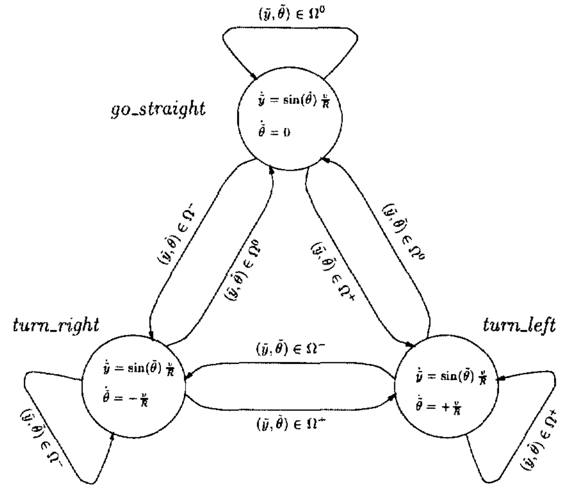

Figure 4: Hybrid model of the closed-loop system.

3.2 Optimal feedback control

In this section, we consider the stability and convergence properties of a piecewise constant feedback control law $\omega(\tilde{y},\tilde{\theta})$ deduced from the optimal synthesis. Due to the coupling of continuous and discrete phenomena, the hybrid systems framework is well appropriate to study the closed–loop system [1]. The hybrid control is characterized by three modes (see figure 4).

$$\begin{array}{lll} \bullet \quad \textit{go\_straight}, & \text{where } \omega = 0; & \\ \bullet \quad \textit{turn\_right}, & \text{where } \omega = -V/R; & (17) \\ \bullet \quad \textit{turn\_left}, & \text{where } \omega = +V/R. & \end{array}$$

selected, according to Proposition 1, as follows:

$$\begin{array}{l} \textit{go\_straight} \text{ if } (\tilde{y},\tilde{\theta}) \in \Omega^0 = \mathbf{sr} \cup \mathbf{sl} \cup \mathbf{\tilde{o}}; \\ \textit{turn\_right} \text{ if } (\tilde{y},\tilde{\theta}) \in \Omega^- = \mathbf{r} \cup \mathbf{rsr} \cup \mathbf{rsl} \cup \mathbf{rl}^{(1)} \cup \mathbf{rl}^{(2)}; \\ \textit{turn\_left} \text{ if } (\tilde{y},\tilde{\theta}) \in \Omega^+ = \mathbf{l} \cup \mathbf{lsr} \cup \mathbf{lsl} \cup \mathbf{lr}^{(1)} \cup \mathbf{lr}^{(2)}; \end{array} \quad (18)$$

The hybrid system remains in a given mode $q \in \{\textit{go\_straight}, \textit{turn\_left}, \textit{turn\_right}\}$ as long as all the guard conditions $(\tilde{y},\tilde{\theta}) \in \Omega$ located on the outgoing arcs are false; when one of them becomes true, the hybrid system switches to the corresponding new mode. As the sets Ω^0,Ω^+ and Ω^- make a partition of the domain $\mathcal{G}_{(\tilde{y},\tilde{\theta})}$, the hybrid automaton is deterministic.

Proposition 2 *The origin of the reduced state space $(\tilde{y},\tilde{\theta})$ is an asymptotically stable equilibrium point of system (11) under feedback (17-18). Convergence is achieved for any initial state $(\tilde{y}_0,\tilde{\theta}_0) \in \mathcal{G}_{(\tilde{y},\tilde{\theta})}$, that is for any initial vehicle configuration (x_0,y_0,θ_0).*

Proof. The convergence is implied by the previous discussion. The proof of stability is based on a direct application of Lyapunov's definition in the reduced space. Let $\xi \stackrel{def}{=} (\tilde{y},\tilde{\theta})^T$, we want to prove that $\forall\epsilon, \exists\delta / \ \|\xi_0\| < \delta \rightarrow \|\phi_{\xi_0}(t)\| < \epsilon, \forall t > 0$, where $\phi_{\xi_0}(t)$ is the trajectory of the closed loop system from $\xi(0) = \xi_0$. Let $W(\xi) = \tilde{y}^2 + \tilde{\theta}^2$, its time derivative along the trajectories of the system is $\dot{W} = \frac{2V}{R}\tilde{y}\sin\tilde{\theta} + \tilde{\theta}\omega$. In particular, in the unit disk $W(\xi) < 1$, we have:

$$\dot{W}(\tilde{y},\tilde{\theta}) = \left\{ \begin{array}{l} (\tilde{y}|\sin(\tilde{\theta})| - |\tilde{\theta}|)\frac{2V}{R} < 0, \\ \quad \left[(\tilde{y},\tilde{\theta}) \in \mathbf{rl}^{(1)} \wedge \tilde{\theta} > 0\right] \vee (\tilde{y},\tilde{\theta}) \in \mathbf{r} \\ (-\tilde{y}|\sin(\tilde{\theta}|) + |\tilde{\theta}|)\frac{2V}{R} \geq 0, \\ \quad (\tilde{y},\tilde{\theta}) \in \mathbf{rl}^{(1)} \wedge \tilde{\theta} \leq 0 \\ (-\tilde{y}|\sin(\tilde{\theta}|) - |\tilde{\theta}|)\frac{2V}{R} < 0, \\ \quad \left[(\tilde{y},\tilde{\theta}) \in \mathbf{lr}^{(1)} \wedge \tilde{\theta} < 0\right] \vee (\tilde{y},\tilde{\theta}) \in \mathbf{l} \\ (\tilde{y}|\sin(\tilde{\theta})| + |\tilde{\theta}|)\frac{2V}{R} \geq 0, \\ \quad (\tilde{y},\tilde{\theta}) \in \mathbf{lr}^{(1)} \wedge \tilde{\theta} \geq 0 \end{array} \right. \quad (19)$$

For any $\beta < \frac{1}{2}$, integrating (19) we obtain

$$\sup_{\|\xi_0\| \leq \beta} \sup_{t \in \mathbb{R}^+} W(\phi_{\xi_0}(t)) = \sup_{t \in \mathbb{R}^+} W(\phi_{(\pm\beta,0)}(t)) \\ = \frac{\beta^2}{4} + \arccos^2(1 - \frac{\beta}{2}) = \bar{W}(\beta) < 1$$

Hence, $\forall\epsilon > 0$, choosing either $\delta = \frac{1}{2}$ if $\epsilon \geq [\bar{W}(\frac{1}{2})]^{\frac{1}{2}}$, or $\delta = \bar{W}^{-1}(\epsilon^2)$ otherwise, for any ξ_0 with $\|\xi_0\| < \delta$ we obtain $\|\phi_{\xi_0}(t))\| < \epsilon < 1$.

4 Experimental results

Experimental tests have been conducted with a wheeled vehicle (TRC's "Labmate"). We have fixed $R = 25cm$ and $V = 5cm/sec$. Information on the vehicle position and orientation is obtained by processing odometric information along with angular measurements given by a ladar sensor (Siman's "Robosense") mounted on the vehicle. Implementations of switching control signals such as (18), on physical plants with such nonidealities, are doomed to produce vibratory phenomena known as "chattering". An example of this behaviour is reported in figure 5, referring to an experiment where raw data from the ladar sensor where fed directly to the control law (18). The optimal connecting path is in this case of type **rl**, but its execution is rather imprecise. We have smoothed the control by introducing a thin "boundary layer" around the curves in state space where discontinuity of the control arises. As a net effect on the global preformance, we have that asymptotic stability claims reduce to uniform ultimate boundedness of trajectories. The claim of optimality of trajectories also fails in this case, but a reasonably good approximation of the optimum can be obtained, as shown in the experimental results below (figure 6).

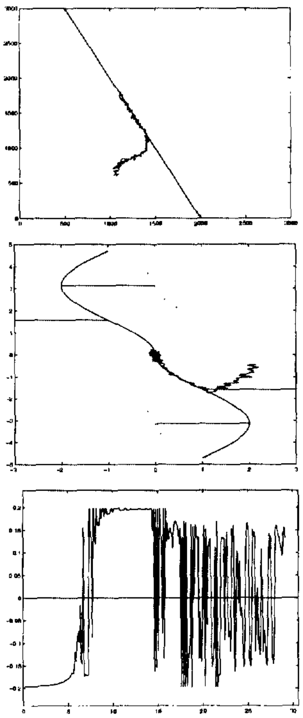

Figure 5: *Experiment 1: (Chattering control) From top down: reference route and vehicle position during execution of the control law; the same positional data in the reduced state space; control input vs. time.*

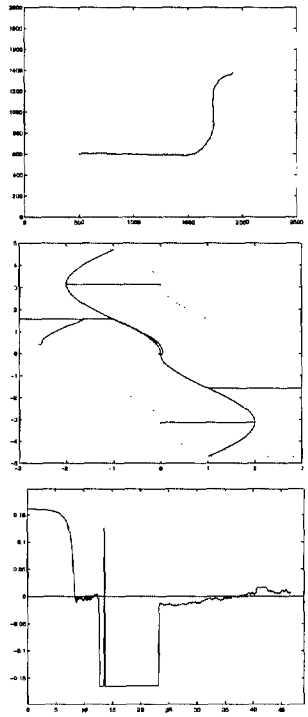

Figure 6: *Experiment 2: Same sequence as before with smoothed control.*

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