A Group-Theoretic Characterization of Quantized Control Systems

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Abstract— **In this paper we consider the reachability problem for quantized control systems, i.e. systems that take inputs from a finite set of symbols. Previous work addressed this problem for linear systems and for some specific classes of nonlinear driftless systems. In this paper we attack the study of more general nonlinear systems. To do so we find it useful to pose the problem in more abstract terms, and make use of the wealth of tools available in group theory, which enables us to proceed in our agenda of better understanding effects of quantization of inputs on dynamic systems.**

I. Introduction

Quantized control systems often represent a proper model to deal with several real-world control systems, among which for instance are applications using switching actuators, or qualitative measurements, or plants where the hardware implementation of the controller loop only admits information transfer with a finite bandwidth.

Several seminal contributions have appeared in recent years on such problems, including those of [4], [5], [2], [6]. In our previous work we have considered in some detail the analysis of the reachable set and the synthesis of open-loop controls. A typical question arising under this regard is whether, for a given set of input symbols, the recahble set is everywhere dense or not, and if not, if there are useful structires in the reachable set, such as e.g. a lattice structure.

These questions, which have a direct bearing on steering systems from one state to another and indirectly also affect stabilization policies, have been answered in [7], [1] for a particular class of systems, i.e. nonlinear driftless systems in chained form. Although this class is rather general and interesting in applications (most nonholonomic systems can be written in such form by a suitable feedback and state diffeomorphism), in this paper we aim at generalizing the approach. In particular, we will focus here on nonlinear driftless systems which are not in chained form, and are subject to quantized inputs. Two examples will be considered for illustration: the case of a rolling polyhedron (which is the quantized counterpart of the plate-ball system, hence is not equivalent to chained form), and the n-trailer vehicle system (for which a feedback transformation to chained form only exists if the control can take continuous values). Our program is to embed these more general problems into the general framework of group actions so as to reduce the basic questions of density/discreteness of reachable sets to the study of normal subgroups, for which a wealth of tools are available form group theory.

The action of sequences of controls can be formalized, under suitable assumptions, as a group action of a set of words. Invertibility of control action is required. In general the set of controls depend on the state and we first stratify the state space by equivalence of control sets. Then we focus on the action of the group on a single equivalence class considering words for which the equivalence class is invariant. Most of literature in group action theory is dedicated to the case of Lie groups, but in our case the discreteness of control sets force us to remain at level of general groups. Orbits for the group action are precisely the reachable sets for the system. We introduce additional assumptions to have homogeneity of the space of orbits. The case of isometries is of particular interest, since in this case we have that the reachable set is formed either by accumulation points or by isolated ones. Then we introduce our main tool: normal subgroups. We show that if isotropy groups coincide along an orbit, then they are normal subgroups and, up to a quotient, we can reduce to a free action.

In general the action of a normal subgroup H can be split in two parts: the first the action of the subgroup H on its (sub)orbit and then the action of the quotient over H over the set of H orbits.

This splitting can be viewed as a base-fiber splitting of the state space and it is natural to describe nonholonomic behavior. For the polyhedron example (as well as for isometries groups over $\mathbb{R}^3$) the set of translations (obtained by rotation along edges) is a normal subgroup and the corresponding fibration was used in [3] to detect density of reachable sets. Another important example is that of chain systems in sampled integrated form, see [7], [1]. Also in this case a complete classification of topologies of reachable sets was obtained through a natural base-fiber reduction.

II. Defintions and Fundamental Assumptions

We begin with describing a quantized control system in the language of the theory of groups.

A discrete time–invariant quantized control system is a 4–tuple $(\mathcal{Q}, \mathcal{U}, \mathcal{A}, \Omega)$ with $\mathcal{Q}$ denoting the configuration set, $\mathcal{U}$ a set of admissible input symbols, $\mathcal{A}$ a state–transition map $\mathcal{A} : \mathcal{Q} \times \mathcal{U} \to \mathcal{Q}$. Notice that in general $\mathcal{U}$ is to be considered as state-dependent, hence $\mathcal{A}(q, u)$ is well defined if $u \in \mathcal{U}$ is admissible for $q \in \mathcal{Q}$. Moreover if u_1 is admissible for q and u_2 is admissible for $\mathcal{A}(q, u_1)$ then we say that u_1 and u_2 are concatenable from q and denote u_1u_2 the concatenation of u_1 and u_2. By recursion we define an "admissible input stream" from a point $q \in \mathcal{U}$ to be the concatenation of concatenable symbols in $\mathcal{U}$ from $q \in \mathcal{Q}$ and denote by Ω_q the set of admissible input streams from q and $\Omega = \cup_{q\in\mathcal{Q}}\Omega_q$.

Next we will give more structure to our sets in order to have a suitable definition for the transition map $\mathcal{A}$.

Consider the multivalued function $\phi : \mathcal{Q} \to \mathcal{U}$ where $\phi(q) = \mathcal{U}_q \subset \mathcal{U}$ is the set of admissible inputs at q. Consider the equivalence relation on $\mathcal{Q}$ given by $q_1 \sim q_2$ iff $\phi(q_1) = \phi(q_2)$, and denote $\mathcal{Q}/\phi$ the set of equivalence classes, $[q]$ the equivalence class of q.

H0 We assume that each equivalence class is a connected submanifold of $\mathcal{Q}$.
Thus we have that the map $\mathcal{A}$ is well defined on each of the product $[q_1] \times \mathcal{U}_{q_1}$:

$$\mathcal{A} : [q_1] \times \mathcal{U}_{q_1} \to \mathcal{Q}$$

where $\mathcal{A}(q_2, u)$ is the state that the system reaches from $q_2 \in [q_1]$ under $u \in \mathcal{U}_{q_1}$. Notice that, in general $\mathcal{A}(q_2, u) \not\sim q_2$.

Consider the following conditions:

H1 $\forall q_1 \sim q_2$ **and** $\forall u \in \mathcal{U}_{q_1} (= \mathcal{U}_{q_2})$ $\mathcal{A}(q_1, u) \sim \mathcal{A}(q_2, u)$**.**

Condition **H1** is referred to as the compatibility of the map $\mathcal{A}$ with respect to the equivalence relation $\sim$. Condition **H1** implies that if $q_1 \sim q_2$ then $\Omega_{q_1} = \Omega_{q_2}$. Hence we can define the map, which, by slight abuse of notation, is also denoted by $\mathcal{A}$,

$$\mathcal{A} : [q_1] \times \Omega_{q_1} \to \mathcal{Q}$$

where

$$\mathcal{A}(q_2, \omega) = \mathcal{A}(\mathcal{A}(\cdots \mathcal{A}(q_2, u_N), \cdots, u_2), u_1),$$

defines the state that the system reaches from $q_2 \in [q_1]$ under $\omega = u_1 \cdots u_N \in \Omega_{q_2} (= \Omega_{q_1})$. Moreover $\mathcal{A}(q_1, \omega) \sim \mathcal{A}(q_2, \omega)$, that is the new map $\mathcal{A}$ is compatible with the equivalence relation $\sim$.

Denote by $\widetilde{\Omega}_q = \{\omega \in \Omega_q : \mathcal{A}(q, \omega) \in [q]\}$ the subset of input streams steering the system back to the same equivalence class of the initial point. By **H1** $\forall q_1 \sim q_2$ $\widetilde{\Omega}_{q_1} = \widetilde{\Omega}_{q_2}$.

We assume also the following condition:

H2 $\forall q \in \mathcal{Q}$**,** $\widetilde{\Omega}_q$ **is a group with the concatenation law, neutral element** e**, and inverse** $\bar{\omega}$ **for all** $\omega \in \widetilde{\Omega}_q$**. Moreover we assume that** $\mathcal{A}(q, e) = q$**.**

Clearly, by **H2**, $\mathcal{A}(\mathcal{A}(q, \omega), \bar{\omega}) = \mathcal{A}(q, \omega\bar{\omega}) = \mathcal{A}(q, e) = q$.

Finally, for all equivalence classes, we have an action of the group $\widetilde{\Omega}_q$ on $[q]$ with transition map $\mathcal{A}$.

The following condition

H3 for all pairs $[q_1], [q_2] \in \mathcal{Q}/\phi$ **there exists** $\omega \in \Omega_{q_1}$ **such that** $\mathcal{A}(\cdot, \omega) : [q_1] \to [q_2]$ **is an homeomorphism and the map** $h : \widetilde{\Omega}_{q_1} \to \widetilde{\Omega}_{q_2}$**, given by** $\omega_1 \in \widetilde{\Omega}_{q_1} \mapsto \bar{\omega}\omega_1\omega = \omega_2 \in \widetilde{\Omega}_{q_2}$**, is a group isomorphism.**

implies that the groups $\widetilde{\Omega}_q$ are conjugate and the map $\mathcal{A}(\cdot, \omega)$ is a h–homeomorphism.

This means that we can study the action of one of the groups $\widetilde{\Omega}_q$ on the equivalence class $[q]$ because for the other equivalence classes we will have the same behavior of the action.

From now on we will then assume conditions **H0,. . . ,H3** and restict ourselves to an action of a group Ω on the connected manifold $\mathcal{Q}$:

$$\mathcal{A} : \mathcal{Q} \times \Omega \to \mathcal{Q}.$$

Observe that $a : \Omega \to \mathcal{S}$ from Ω in the set of mappings $\mathcal{Q}^{\mathcal{Q}}$ of $\mathcal{Q}$ into itself, where $a(\omega) = \mathcal{A}_\omega$ and $\mathcal{A}_\omega(q) = \mathcal{A}(q, \omega)$, is a group homomorphism.

We say that the action of $\mathcal{A}$ is effective if $\ker a = \{e\}$. If the action is effective we have that $\forall \omega \in \Omega, \omega \neq e$, there exists $q \in \mathcal{Q}$, such that $\mathcal{A}(q, \omega) \neq q$. If $\ker a = N$ then we have that the action of Ω/N on $\mathcal{Q}$ is effective hence, up to quotient, we can assume that we have an effective action of Ω on $\mathcal{Q}$.

With this assumption we then have that, if $\mathcal{A}(\mathcal{A}(q, \omega_1), \omega_2) = q$ then $\omega_1\omega_2$ is identified with the identity element e, hence ω_1 is identified with $\bar{\omega}_2$.

Observe that, even if the action is effective we can have fixed points, i.e. points $q \in \mathcal{Q}$ such that $\mathcal{A}(q, \omega) = q$ for all $\omega \in \Omega$.

We are interested in the analysis of the reachable set of a quantized control system. In our framework it means that we shall analyse, from a topological and measure point of view, the orbits of $a(\Omega)$ from a point $q \in \mathcal{Q}$ that is $\mathcal{R}_q = \{\mathcal{A}_\omega(q) : \omega \in \Omega\}$. The set of orbits is given by the quotient $\mathcal{Q}/a(\Omega)$.

We say that an action is transitive if $\forall q_1, q_2 \in \mathcal{Q}$ there exists $\omega \in \Omega$ such that $q_2 = \mathcal{A}(q_1, \omega)$. Since Ω is a discrete group we never have a transitive action. Clearly the action is always transitive on one orbit $\mathcal{R}_q$ and we say that $\mathcal{R}_q$ is a homogeneous Ω–set.

Example: consider a polyhedron rolling on a plane around the edges. The position of the polyhedron is determined assigning the face that lyes on the plane, the position and orientation of this face with respect to a coordinate system on the plane. Thus the state space is given by $\mathcal{Q} = \mathcal{F} \times \mathbb{R}^2 \times S^1$, *where* $\mathcal{F} = \{F_1, \ldots, F_n\}$ *is the set of faces of the polyhedron.*

Fix $q = (F_i, \bar{x}, \bar{\theta})$, *then the possible controls are determined by the edges of the face* F_i. *Indeed the possible actions are rotations around one of such edges until a face of the polyhedron adjacent to* F_i *lyes on the plane. Therefore, if we denote by* $\{F_j : j \in J_i\}$, $J_i \subset \{1, \ldots, n\}$, *all adjacent faces to* F_i, *we can describe the set of inputs admissible at* q *as* $\mathcal{U}_q = \{F_j : j \in J_i\}$. *Then* Ω_q *is the set of words* $F_{i_1}, \ldots, F_{i_m}$, $m \in \mathbb{N}$, *such that* $i_1 \in J_i$ *and* $i_j \in J_{i_{j-1}}$, $j = 2, \ldots, m$.

Each equivalence class $[q]$, $q = (F_i, \bar{x}, \bar{\theta})$, *is given by* $\{(F_i, x, \theta) : x \in \mathbb{R}^2, \theta \in S^1\}$.

Assumptions **H0-H1** *are obviously verified. Notice that* $\widetilde{\Omega}_q$, $q = (F_i, \bar{x}, \bar{\theta})$, *is formed by the words* $F_{i_1}, \ldots, F_{i_m} \in \Omega_q$ *such that* $i_m = i$. *Defining the neutral element as the empty word, since every action* F_j, $j \in J_i$, *is invertible, we get that* **H2** *is also verified.*

Now given two equivalence classes $[q_1] = (F_{i_1}, \cdot, \cdot)$ *and* $[q_2] = (F_{i_2}, \cdot, \cdot)$ *let* ω *be any word steering the polyhedron from* $[q_1]$ *to* $[q_2]$. *Then the map* $\mathcal{A}(\cdot, \omega)$ *is clearly an homeomorphism since it is a translation on* $\mathbb{R}^2 \times S^1$. *Moreover the corresponding map* h *is a group isomorphism so* **H3** *holds true. We thus can fix some* q *and study the action of the group* $\Omega = \widetilde{\Omega}_q$ *on* $\mathcal{Q} = [q] \simeq \mathbb{R}^2 \times S^1$. *From previous works ([3]) we have that* $a(\Omega)$ *is a subgroup of the isometries of* $\mathbb{R}^2$ ($\mathcal{A}_\omega(\bar{x}, \bar{\theta}) \mapsto (x + R(\bar{\theta})t, \bar{\theta} + \psi)$ *where* ψ, t *depend on* ω *and* $R(\bar{\theta})$ *is the matrix of plane rotation of angle* $\bar{\theta}$). *In general we have that* $\Omega \neq \{e\}$, *i.e. there may exists an element* $\omega \in \Omega$, $\omega \neq e$ *such that* $\forall q \in \mathcal{Q}$, $\mathcal{A}_\omega(q) = e$. *Therefore we restrict ourselves to the action of* $\Omega/ker(a)$ *on* $\mathcal{Q}$.

We denote by Ω^q, $\Omega^q = \{\omega \in \Omega : \mathcal{A}_\omega(q) = q\}$, the isotropy group for q that is the subgroup of Ω which fixes the point q. We say that the action is free if $\Omega^q = \{e\}$, $\forall q \in \mathcal{Q}$.

H4 $\forall q_1, q_2 \in \mathcal{Q}$ **with** $q_2 \in \mathcal{R}_{q_1}$ $\Omega^{q_2} = \Omega^{q_1}$.

Proposition 1: If **H4** holds then for every $q \in \mathcal{Q}$, Ω^q is a normal subgroup of Ω and Ω/Ω^q acts freely and transitively on the orbit $\mathcal{R}_q$. We say that $\mathcal{R}_q$ is a homogeneous principal Ω–set.

Proof: Fix $q_1 \in \mathcal{Q}$, $\tilde{\omega} \in \Omega^{q_1}$ and $\omega \in \Omega$. We need to show that $\omega\tilde{\omega}\bar{\omega} \in \Omega^{q_1}$. Let $q_2 = \mathcal{A}_\omega(q_1) \in \mathcal{R}_{q_1}$, then

$$\begin{aligned} &\mathcal{A}_{\omega\tilde{\omega}\bar{\omega}}(q_2) = \\ &\mathcal{A}_{\omega\tilde{\omega}\bar{\omega}}\mathcal{A}_\omega(q_1) = \\ &\mathcal{A}_{\omega\tilde{\omega}\bar{\omega}\omega}(q_1) = \\ &\mathcal{A}_{\omega\tilde{\omega}}(q_1) = \\ &\mathcal{A}_\omega\mathcal{A}_{\tilde{\omega}}(q_1) = \\ &\mathcal{A}_\omega(q_1) = q_2, \end{aligned}$$

hence $\omega\tilde{\omega}\bar{\omega} \in \Omega^{q_2} = \Omega^{q_1}$. ■

Thus if **H4** holds then it is not restrictive to assume that $\Omega^q = \{e\}$ hence that $\mathcal{R}_q$ is a homogeneous Ω–space.

If we have more than one orbit we would like the structure of different orbits to be always the same, from a qualitative point of view. This is guaranteed if we assume

H5 For all $q_1, q_2 \in \mathcal{Q}$ there exists a homeomorphism $\varphi : \mathcal{Q} \to \mathcal{Q}$, $\varphi(q_1) = q_2$ such that for every $\omega \in \Omega$ we have $\varphi \mathcal{A}_\omega = \mathcal{A}_\omega \varphi$.

If **H5** holds we get that φ establishes a bijection between $\mathcal{R}_{q_1}$ and $\mathcal{R}_{q_2}$. Moreover the two reachable sets have the same topological properties.

We assume a distance d to be defined on $\mathcal{Q}$ and the following

H6 $\forall \omega \in \Omega$, $\mathcal{A}_\omega$ **is an isometry.**

This means that $\forall q_1, q_2 \in \mathcal{Q}$, $d(\mathcal{A}_\omega(q_1), \mathcal{A}_\omega(q_2)) = d(q_1, q_2)$. Therefore the assumption **H6** implies (see Theorem 1 of [1]) that $\mathcal{R}_q$ is comprised either only of accumulation points or only of isolated points.

From now on we will assume **H6**, fix one point $\bar{q} \in \mathcal{Q}$ and restrict ourselves to the analysis of the orbit $\mathcal{R}_{\bar{q}}$.

Example(continue) In the polyhedron example we have that the isotropy group $\Omega^q = \{e\}$, because $a(\Omega)$ are isometries (hence **H6** *holds true) and if ω fix a point then it fixes all points, hence $\omega \in \ker(a)$. Moreover we consider tha action of $a(\Omega)/\ker(a)$. Therefore assumption* **H4** *is verified and the action of $\Omega/\ker(a)$ on the orbits $\mathcal{R}_q$ is free and transitive.*

We also have that the orbits are isometric. Indeed consider q_1, q_2 any two points of $\mathcal{Q}$ and φ a rotation of $\mathbb{R}^2 \times S^1$ around x_1 followed by a translation of $\mathbb{R}^2$ such that $\varphi(q_1) = q_2$. Then φ is an isometry and satisfies **H5**. *We thus can restrict our study to a single orbit.*

A simple example is given by the manipulation of a cube with side of length ℓ. Fix a face, say F_1, and consider the orbit through $(x, \theta) \in \mathbb{R}^2 \times S^1$. Then we can reach all points with first component on a square lattice of side ℓ and orientation of type $\theta + k\pi/2$.

III. Subgroup actions and base–fiber decompositions

Let $H \subset \Omega$ be a subgroup. Then $a(H) \subset a(\Omega)$ is a subgroup. Indeed if $\omega_1, \omega_2 \in H$ then $\mathcal{A}_{\omega_1}, \mathcal{A}_{\omega_2} \in a(H)$ and $\mathcal{A}_{\omega_1}\mathcal{A}_{\omega_2} = \mathcal{A}_{\omega_1\omega_2} \in a(H)$. Therefore we could consider the orbit of q under the action of H. Denoting $\mathcal{R}_q^H$ the orbit under the action of the elements of a subgroup H we clearly have $\mathcal{R}_q^H \subset \mathcal{R}_q$. In particular we notice that Ω^q, the isotropy group of q is a subgroup of Ω and $\mathcal{R}_q^{\Omega^q} = \{q\}$.

If H is a normal subgroup of Ω then, by definition, $\forall \omega \in \Omega$, $\omega H \bar{\omega} = H$ and Ω/H is a group. As $\mathcal{R}_q$ is a homogeneous Ω–space so $\mathcal{R}_q^H$ is a homogeneous H–space. This approach allow us to first study the action of the normal subgroup H on q and, second, the action of Ω/H on the set of orbits $\{\mathcal{R}_{q'}^H : q' \in \mathcal{R}_q\}$.

We observe that if $q_2 = \mathcal{A}_h(q_2)$

$$\mathcal{A}_\omega(q_2) = \mathcal{A}_{\omega h}(q_1) = \mathcal{A}_{h'\omega}(q_1)$$

where the last equation is obtained by applying the definition of normal subgroup. Then

$$\mathcal{A}_\omega(q_2) = \mathcal{A}_{h'}\mathcal{A}_\omega(q_1),$$

i.e. the transition from two different points of the same H–orbit are mapped to the same H–orbit. But, since h is in general different from h', the actions do not commute (this is important from an algorithmic point of view). In order to have commuting actions we could choose $H = [\Omega, \Omega]$ the subgroup of commutators.

Let us describe the link between normal subgroups and base–fiber decompositions.

Definition 1: Let H and G be two groups and $\tau : G \to \mathrm{Aut}(H)$, $\tau(g) = \tau_g$ a homomorphism of G into the group of automorphism group of H, i.e. $\tau_g : H \to H$. The set $H \times G$ with the composition:

$$\begin{array}{ccc} (H \times G) \times (H \times G) & \to & (H \times G) \\ ((h, g), (h', g')) & \mapsto & (h\tau_g(h'), gg') \end{array}$$

is called the external semi–direct product of G by H relative to τ and is denoted by $H \times_\tau G$.

Proposition 2: The external semi–direct product $H \times_\tau G$ is a group. The mappings $i : H \to H \times_\tau G$, $i(h) = (h, e)$, $p : H \times_\tau G \to G$, $p(h, g) = g$, and $s : G \to H \times_\tau G$, $s(g) = (e, g)$ are group homomorphism and s is a section, i.e. $p \circ s$ is the identical mapping from G to G.

Proposition 3: If $\Omega = H \times_\tau K$ then we can decompose $\mathcal{R}_q$ into base $\mathcal{R}_q^H$ and fiber $\mathcal{R}_q^G$.

The following proposition gives a conditions for the existence of semi–direct decomposition.

Proposition 4: Let Ω be a group, $H \subset \Omega$ a normal subgroup and $G \subset \Omega$ a subgroup such that $H \cap G = \{e\}$ and $HG = \{hg : h \in H, g \in G\} = \Omega$. Let $\tau : G \to \mathrm{Aut}(H)$, $\tau(g) = \tau_g$ with $\tau_g(h) = gh\bar{g} \in H$. Then the map $(h, g) \mapsto hg$ is an isomomorphism of $H \times_\tau G$ onto Ω.

Example (continue). In the example of the polyedron rolling on a plane, each element $\omega \in \Omega$ *correponds to a rototranslation hence can be written as a pair* $(t, \theta) \in \mathbb{R}^2 \times S^1$ *with the composition rule* $\omega'\omega = (t, \theta)(t', \theta') = (t + e^{i\theta}t', \theta + \theta')$*. Hence there is an isomorphism of* $H \times_\tau G$ *onto* Ω *where* H *is the subgroup of translations, a normal subgroup of* Ω *and* $G \subset S^1$ *is a group of rototranslation that is finite in case of discrete reachable sets and infinite otherwise.*

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