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Path Tracking Control for Dubin's Cars

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Abstract

The problem of driving a Dubin's car along a given path is considered. In order to model a realistic road-following problem, the car is supposed to move forward only and to have bounds on the turning radius (Dubin's car). We propose a discontinuous control scheme on the angular velocity of the vehicle, based on the theory of sliding modes, that achieves the goal of tracking an unknown path relying on measurements of the current distance from the path and of the heading angle error.

1 Introduction

The literature on planning and control techniques for nonholonomic vehicles has grown extensive in the recent few years. The planning problem for nonholomic vehicles requires an approach based on a mix of techniques from conventional, holonomic planning and nonlinear systems theory. Besides the kinematic constraints imposed by nonholonomy, most often the additional constraint that the radius of curvature of the paths of the vehicle are lower bounded must be considered. It has been shown that the kinematic model of a car that can drive both forwards and backwards with bounded curvature (allowing cusps in the path), is locally controllable. A car that can only move forwards with curvature bounds (the "Dubin's car") is still controllable, although not locally. For this latter type of vehicle, Dubin [5] studied the shortest paths joining two arbitrary configurations (these are compound of line segments and arcs of circles of minimum radius). Reeds and Shepp [8] extended Dubin's results to a car that can reverse its motion. The control problem is particularly challenging for nonholonomic systems, due to a theorem of Brockett [3] that bars the possibility of stabilizing a nonholonomic vehicle about a nonsingular configuration by any continuous time-invariant static feedback. Non-smooth (see e.g. Sordalen and Egeland, [14], Aicardi *et al.*, [1], Astolfi, [2], Guldner and Utkin, [6]), time-varying (viz. Samson, [10], McCloskey and Murray, [7], Sampei *et al.*, [9]), and dynamic extension algorithms (see e.g. DeLuca and DiBenedetto, [4]), have been proposed to face the point-stabilization problem. For nonholonomic systems, the problem of tracking a trajectory or a path is simpler in principle than stabilizing to a point. Here, by "path" we refer to a curve (with some regularity requirements) in the plane were the robot moves; while a "trajectory" is a path with an associated time law. For an example of trajectory tracking controllers, see e.g. Walsh *et al.*, [17]; and Sordalen and Canudas de Wit, [13], Sarkar *et al.*, [11] for path tracking controllers. In most part of the nonholonomic vehicle control literature, however, curvature bounds on the trajectories resulting from application of the control laws have not been considered. The work of Souères and Laumond [15], who mapped the whole configuration space of a Reeds and Shepp car in the optimal trajectories to a given goal configuration, can be used to build a stabilizing feedback law with bounded-curvature paths. In this paper we consider the design of a control law for path tracking by a Dubin's car. The restriction that the vehicle only moves forward is motivated by the fact that, in practical road-following problems, vehicles maintain a positive velocity. We therefore assume that the forward velocity is given, and are only concerned with lateral stabilization to the path. The path shape is free (under some mild regularity restrictions), and it is not assumed that it is known a priori to the controller. We assume also that the only information available to the controller is the vehicle's lateral distance from the path, its heading angle error, and the sign of the curvature of the reference path. As a result, we propose a variable-structure control law for vehicle orientation, that stabilizes the vehicle on the given path. The controller is designed according to sliding-mode techniques, and is therefore discontinuous in time. However, in practice it can be implemented in a smoothed version, that eliminates chattering and maintains good performance.

2 Problem formulation

Let the reference path $\gamma \subset \mathbb{R}^2$ be described as the trace of the parametrized curve

$$\mathrm{g}(s) = [\ x(s),\ y(s)\ ]^T \quad \text{with } s \in (0,1),$$

with the natural orientation induced by increasing s. The following restrictions are imposed on the path:

A) $\mathrm{g}(s)$ has continuous first derivative $\mathrm{g}'(s)$ in $(0,1)$. The second derivative $\mathrm{g}''(s)$ has only a finite number of discontinuity points in $(0,1)$, and changes its sign only a finite number of times in $(0,1)$.

B) Let r denote the minimum turning radius of the vehicle, and $R(s)$ the radius of curvature of $\mathrm{g}(s)$. Set $R(s)$ to $+\infty$ at the inflexional points of $\mathrm{g}(s)$ and at the discontinuity points of g''. Otherwise, set

$$\frac{1}{R(s)} = \frac{\hat{x}'\hat{y}'' - \hat{y}'\hat{x}''}{(\hat{x}'^2 + \hat{y}'^2)^{3/2}}.$$

Assume that

$$|R(s)| \geq r, \quad s \in (0,1). \qquad (1)$$

C) Consider an open neighbourhood of the path of radius r,

$$\mathcal{T}_\gamma = \{\mathrm{x} \in \mathbb{R}^2 : \exists s \in (0,1), \|\mathrm{x} - \mathrm{g}(s)\| < r\} \subset \mathbb{R}^2$$

where $\|\cdot\|$ denotes the euclidean norm, and assume that for all $\mathrm{x} \in \mathcal{T}_\gamma$ there exists a unique point on g, say at $\bar{s} \in (0,1)$, such that

$$\|\mathrm{x} - \mathrm{g}(s)\| > \|\mathrm{x} - \mathrm{g}(\bar{s})\|, \quad \forall s \neq \bar{s}.$$

As a consequence of this assumption the path is simple.

Let x, y, θ denote the position and orientation of a unicycle with respect to the world frame. The kinematic equations of the Dubin's car are written as

$$\left\{ \begin{array}{rcl} \dot{x} &=& \cos(\theta)\,u \\ \dot{y} &=& \sin(\theta)\,u \\ \dot{\theta} &=& \omega \end{array} \right., \quad \begin{pmatrix} x(0) \\ y(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ \theta_0 \end{pmatrix}, \qquad (2)$$

where the linear and angular velocity u, ω are subjected to the constraints

$$u = \text{ cost } > 0 \quad \text{and} \quad \left|\frac{\omega}{u}\right| \leq \frac{1}{r}. \qquad (3)$$

We adhere to the assumption on u to be constant, although the generalization to the case $u(t) > 0$ is straightforward. In order to formalize and solve the control task of steering ω so as to converge to the desired path and track it (with given velocity u), it is expedient to introduce a different set of coordinates for the state space. The path is embedded in a three-dimensional space, and the canonical frame $\mathcal{S}_T(s)$ associated with $\mathrm{g}(s) = (\hat{x}(s), \hat{y}(s), 0)^T \in \mathbb{R}^3$ is considered. Recall that the canonical frame for a curve is defined by the tangent, the principal normal and the binormal of the curve at each point. In our case, the tangent and principal normal of $\mathcal{S}_T$ remain within the plane where the car moves, while the binormal points upwards or downwards, depending on the local curvature, i.e., on $\mathrm{sign}(R(s))$. Let $\hat{\theta}(s)$ denote the orientation of the tangent of the curve with respect to the x axis of the world frame

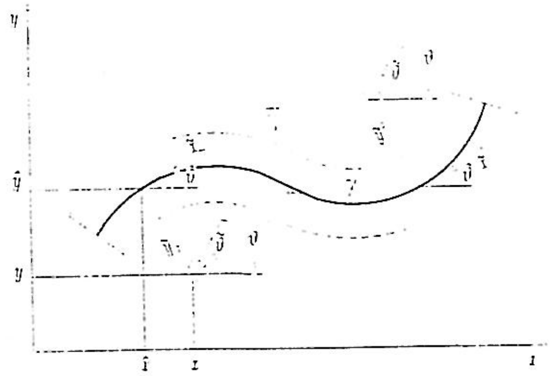

Figure 1: Reference path and coordinates associated with the configurations of the vehicle.

$$\hat{\theta}(s) = \mathrm{atan2}\ (\hat{y}', \hat{x}').$$

Note that by assumption **A)** on $\mathrm{g}(s)$, $\hat{\theta}$ is a continuous function in terms of s. Denote by $(\bar{x}, \bar{y}, \bar{\theta})^T$ the configuration of the vehicle with respect to $\mathcal{S}_T$ (see fig.1). Notice that the positive sense of $\bar{\theta}$ is taken according to the local orientation of the binormal axis. From elementary geometry we get

$$\begin{array}{rcl} \bar{x}(x,y,s) &=& (x - \hat{x}(s))\cos(\hat{\theta}(s)) \\ && +(y - \hat{y}(s))\sin(\hat{\theta}(s)), \end{array} \qquad (4)$$

$$\begin{array}{rcl} \bar{y}(x,y,s) &=& \mathrm{sign}(R(s))\Big[(y - \hat{y}(s))\cos(\hat{\theta}(s)) \\ && -(x - \hat{x}(s))\sin(\hat{\theta}(s))\Big], \end{array} \qquad (5)$$

$$\bar{\theta}(\theta, s) = \mathrm{sign}(R(s))(\theta - \hat{\theta}(s)). \qquad (6)$$

Based on assumption **C)**, it is possible to associate to every point $(x,y)^T$ of the neighborhood $\mathcal{T}_\gamma$ of the path, a unique frame $\mathcal{S}_T(\bar{s})$ with origin in the point of the path closest to $(x,y)^T$. In fact, an application

$\bar{s} : \mathcal{T}_\gamma \subset \mathbb{R}^2 \rightarrow (0, 1) \in \mathbb{R}$ is implicitly defined through (4) as

$$\tilde{x}(x, y, \bar{s}) = 0 \qquad (7)$$

By our assumptions on the path, it also follows that $\bar{s}(x, y)$ is continuous everywhere, and differentiable almost everywhere, on $\mathcal{T}_\gamma$. Consider the change of coordinates

$$\mathcal{M} : \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \rightarrow \begin{pmatrix} \bar{s} \\ \tilde{y}|_{\bar{s}} \\ \tilde{\theta}|_{\bar{s}} \end{pmatrix}, \qquad (8)$$

on the domain $\mathcal{T}_\gamma \times \mathbb{R}$ by means of (7), (5), and (6). The new coordinates $\tilde{y}$ and $\tilde{\theta}$ are the lateral distance from the path and the heading angle error of the vehicle, i.e., they represent a natural choice for describing a road-following task. The change of coordinates is legitimate as it is injective on its domain. Notice however that this change of coordinates not only is not a diffeomorphism (as changes of coordinates are usually in nonlinear control theory), but it is not even continuous. In fact, due to the presence of the term $\text{sign}(R(s))$ in (5), (6), a change of curvature along the path produces a jump of the variables $\tilde{y}, \tilde{\theta}$ to the symmetric point with respect to the origin in the $\tilde{y}, \tilde{\theta}$ plane (see fig.2). The introduction of discontinuous

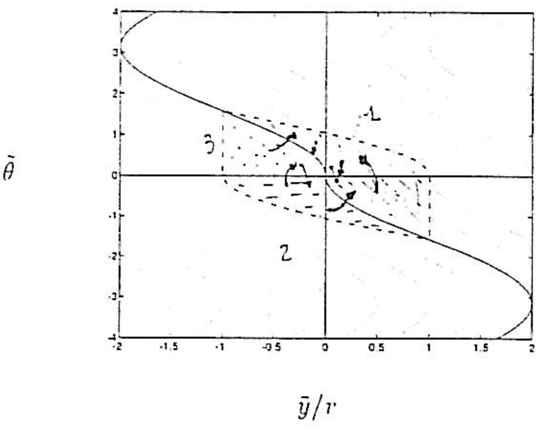

Figure 2: Sliding manifold (solid line), level surfaces of the σ function (dotted lines), and domain of attractiveness in the reduced state space.

changes of coordinates have been used previously in the nonholonomic vehicle literature, e.g. by Astolfi [2] and by Aicardi *et al.* [1]. In the latter work, a new coordinate set (along with an input transformation) allowed authors to design smooth asymptotic point stabilizers without violating Brockett's theorem.

By differentiating (7), (5) and (6), applying the implicit function theorem, the dynamic equations (2) in the new state space (whose coordinates will henceforth be denoted by $(s, \tilde{y}, \tilde{\theta})^T$) become:

$$\left\{ \begin{array}{rcl} \dot{s} & = & \frac{\cos(\tilde{\theta})}{\cos(\tilde{\theta})\hat{x}' + \sin(\tilde{\theta})\hat{y}' - \tilde{y}\hat{\theta}'} u \\ \dot{\tilde{y}} & = & \sin(\tilde{\theta})\, u + 2\delta(R(s))\, \text{sign}(R(s))\tilde{y} \\ \dot{\tilde{\theta}} & = & -\frac{\cos(\tilde{\theta})}{|R(s)| - \tilde{y}} u + \text{sign}(R(s))[\omega \\ & & +2\delta(R(s))\, \text{sign}(R(s))\tilde{\theta}] \end{array} \right. . \qquad (9)$$

Whenever the path is not known a priori, the geometric parameters $\hat{x}'(s)$, $\hat{y}'(s)$, $\hat{\theta}'(s)$, appearing in the first state equation are not available to the controller. It should also be noted that, even if the path were known in advance, computations would be extremely awkward for all but the simplest path shapes. This is one reason why most path tracking controllers in the literature assume that reference paths are comprised of straight lines and circles only.

To overcome this difficulty, we consider a reduced state space $(\tilde{y}, \tilde{\theta})^T$, looking at $R(s)$ as a disturbance, about which the only available information is its sign and the lower bound given in assumption B). The origin of the reduced state space corresponds to a motion of the vehicle along the desired path, with velocity u.

Therefore, our control problem can be formulated as follows:

Problem. Find a feedback control law $\omega(\tilde{y}, \tilde{\theta}, \text{sign}(R), u)$ satisfying the curvature constraint (3), such that, for any initial configuration $(x_0, y_0, \theta_0)^T$ of the vehicle in a suitable neighborhood of the path $\mathcal{D}_\gamma$, $(\tilde{y}, \tilde{\theta})$ converge to zero, irrespective of the bounded unknown disturbances $|R(s)|$.

3 Variable Structure Control

As mentioned in the problem formulation, we look for a controller that makes the path attractive for all states in a region near the path itself.

Consider an open neighborhood $\mathcal{D}_\gamma$ of the path in the reduced state space as

$$\mathcal{D}_\gamma = \left\{ \begin{pmatrix} \tilde{y} \\ \tilde{\theta} \end{pmatrix} : |\tilde{y}| < r, \right.$$

$$\left. -\arccos\left(\frac{1}{2} - \frac{\tilde{y}}{2r}\right) < \tilde{\theta} < \arccos\left(\frac{1}{2} + \frac{\tilde{y}}{2r}\right) \right\}, \qquad (10)$$

Notice that $\mathcal{M}^{-1}((0, 1) \times \mathcal{D}_\gamma) \subset \mathcal{T}_\gamma \times (-\pi/2, \pi/2)$. Observe also that $\mathcal{D}_\gamma$ is symmetric with respect to the origin of the plane $(\tilde{y}, \tilde{\theta})$ (see fig.2). Therefore, states within $\mathcal{D}_\gamma$ remain inside $\mathcal{D}_\gamma$ after any change of the sign of $R(t)$. Since, according to assumption A) in section 2, there are only a finite number of such jumps, in our discussion we consider the evolution of states corresponding to open intervals where $R(t)$ does not change sign, and consider the effects of jumps separately.

Our proposed controller is based on the so-called sliding-mode design technique (see e.g. Utkin [16]). Let us introduce a sliding manifold in the reduced state space as

$$\sigma(\tilde{y}, \tilde{\theta}) = 0, \qquad (11)$$

where

$$\sigma(\tilde{y}, \tilde{\theta}) = -\frac{\tilde{y}}{r} - \operatorname{sign}(\tilde{\theta})(1 - \cos(\tilde{\theta})). \qquad (12)$$

Note that the function $\sigma(\tilde{y}, \tilde{\theta})$ is continuously differentiable once with respect to $\tilde{\theta}$, with $\frac{\partial \sigma}{\partial \tilde{\theta}}|_{(\tilde{y},0)} = 0$.

In the sliding mode control literature, the "equivalent control" is the input signal that causes a motion in the state space on the sliding surface $\sigma = 0$. The equivalent control is found by solving the equation $\dot{\sigma}(t) = 0$ in terms of the unknown control input. By differentiating (12)

$$\dot{\sigma} = -\sin(\tilde{\theta}) \left(\frac{u}{r} + \operatorname{sign}(R(t)) \operatorname{sign}(\tilde{\theta}) \omega + \right. \\ \left. - \operatorname{sign}(\tilde{\theta}) \frac{\cos(\tilde{\theta})}{|R(t)| - \tilde{y}} u \right), \qquad (13)$$

the equivalent control ω_{eq} is derived as

$$\omega_{eq} = -\operatorname{sign}(R(t)) \left(\operatorname{sign}(\tilde{\theta}) - \cos(\tilde{\theta}) \frac{r}{|R| - \tilde{y}} \right) \frac{u}{r}. \qquad (14)$$

Note that ω_{eq} does not satisfy the constraint (3) on the minimum radius of curvature for $\tilde{\theta} < 0$.

The dynamics of motion along the sliding manifold are derived by replacing (14) in (9), as:

$$\begin{cases} \dot{\tilde{y}} = \sin(\tilde{\theta})\, u \\ \dot{\tilde{\theta}} = -\operatorname{sign}(\tilde{\theta}) \frac{u}{r} \end{cases} . \qquad (15)$$

Starting from any initial state $(\tilde{y}_0, \tilde{\theta}_0) \neq (0,0)$ on the sliding manifold, $|\tilde{\theta}(t)|$ monotonically decreases until zero is reached in finite time $|\tilde{\theta}_0| r/u$. From (12), we also get $\tilde{y} = 0$. A sliding regime on $\sigma = 0$ therefore implies convergence of the states to the origin of the reduced state space, hence perfect path tracking.

As already noticed, however, the equivalent control is not feasible by our Dubin's car in the region $\tilde{\theta} < 0$. Therefore, we must design a feasible control ω that guarantees attractivity of the feasible portion of the sliding surface $\sigma = 0, \tilde{\theta} \geq 0$. To this purpose, consider the following control law:

$$\omega = \operatorname{sign}(R(t)) \operatorname{sign}(\sigma) \frac{u}{r}. \qquad (16)$$

The corresponding closed loop equations are written as

$$\begin{cases} \dot{\tilde{y}} = \sin(\tilde{\theta})\, u \\ \dot{\tilde{\theta}} = \left(\operatorname{sign}(\sigma) - \cos(\tilde{\theta}) \frac{r}{|R(t)| - \tilde{y}} \right) \frac{u}{r} \end{cases} , \qquad (17)$$

and, plugging (16) into (13), we get

$$\dot{\sigma} = -\sin(\tilde{\theta}) \left(1 + \operatorname{sign}(\tilde{\theta}) \operatorname{sign}(\sigma) \right. \\ \left. - \operatorname{sign}(\tilde{\theta}) \cos(\tilde{\theta}) \frac{r}{|R(t)| - \tilde{y}} \right) \frac{u}{r}. \qquad (18)$$

The analysis may now be carried over, corresponding to a four-fold partition of the neighbourhood $\mathcal{D}_\gamma$:

Region 1: $\left\{ (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma : \sigma > 0, \text{ and } \tilde{\theta} > 0 \right\}$.

From (12), we get $\tilde{y} < 0$ and hence

$$0 < \frac{r}{|R(t)| - \tilde{y}} < 1.$$

From (17), $\dot{\tilde{\theta}} > 0$ and $\dot{\tilde{y}} > 0$. From (12), $\tilde{\theta} < \frac{\pi}{2}$ holds as long as $\sigma > 0$, hence in this sector it holds

$$\sigma \dot{\sigma} = -\sigma \sin(\tilde{\theta}) \left(2 - \cos(\tilde{\theta}) \frac{r}{|R(t)| - \tilde{y}} \right) \frac{u}{r} < 0.$$

If a change of sign of $R(t)$ occurs the state jumps istantaneously to the symmetric point in region 2.

Region 2: $\left\{ (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma : \sigma < 0, \text{ and } \tilde{\theta} < 0 \right\}$.

From (17), $\dot{\tilde{\theta}} < 0$ and $\dot{\tilde{y}} < 0$. Hence $\tilde{y} < r$ and $\tilde{\theta} > -\frac{\pi}{2}$ holds as long as $\sigma < 0$. Then we have

$$\sigma \dot{\sigma} = -\sigma \sin(\tilde{\theta}) \left(2 + \cos(\tilde{\theta}) \frac{r}{|R(t)| - \tilde{y}} \right) \frac{u}{r} < 0.$$

If a change of sign of $R(t)$ occurs the state jumps istantaneously to the symmetric point in region 1.

Region 3: $\left\{ (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma : \sigma < 0, \text{ and } \tilde{\theta} \geq 0 \right\}$.

Consider the function

$$\Gamma(\tilde{y}, \tilde{\theta}) = -1 - \frac{\tilde{y}}{r} + 2 \cos \tilde{\theta}$$

and observe that, within region 3, $\Gamma(\tilde{y}, \tilde{\theta}) > 0 \Leftrightarrow (\tilde{y}, \tilde{\theta}) \in \mathcal{D}_\gamma$. Along the state trajectories

$$\dot{\Gamma} = \sin(\tilde{\theta}) \frac{u}{r} + 2 \sin(\tilde{\theta}) \cos(\tilde{\theta}) \frac{u}{|R(t)| - \tilde{y}} > 0$$

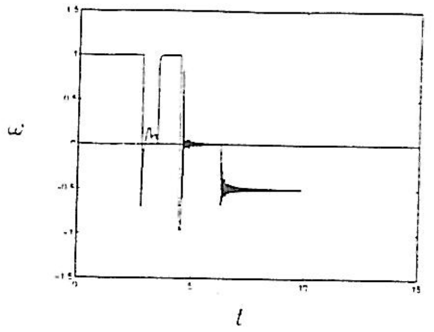

Figure 4: Input signals $\omega(t)$.

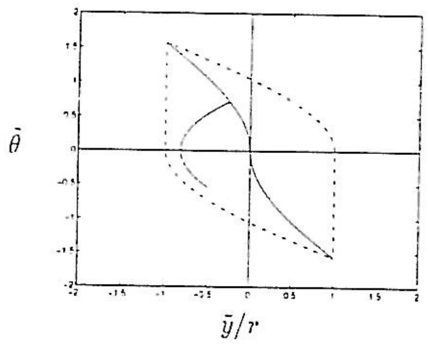

Figure 5: Trajectory on the reduced state space $\bar{y}$, $\bar{\theta}$, domain of convergence and sliding manifold $\sigma = 0$.

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