Workshop on New frontiers of Robotics -
Interdep. Research Center "E. Piaggio"
June 21-22, 2012 - Pisa (Italy)

BOOLEAN CONSENSUS FOR SOCIETIES OF ROBOTS

Adriano Fagiolini

DIEETCAM, College of Engineering, University of Palermo
Interdep. Research Center "E. Piaggio", University of Pisa

WHAT IS IT?

- *A society is a collection of individuals with different levels of autonomy, hierarchical organization and interaction.*

WHAT IS IT?

- *A society is a collection of individuals with different levels of autonomy, hierarchical organization and interaction.*

- A "Society of Robots" (SoR) is a complex system made of:
  - *Many,*
  - *Fast/Slow,*
  - *Different,*
  - *Autonomous,*
  - *Uncoordinated,*
  - *(Fairly) Competing through rules,*
  - *Joining in and quitting at will,*
  - *Mostly well-behaved...* robots.

IMPOSSIBLE?

- Societies in nature

IMPOSSIBLE?

- Societies in nature

IMPOSSIBLE?

- Societies in nature

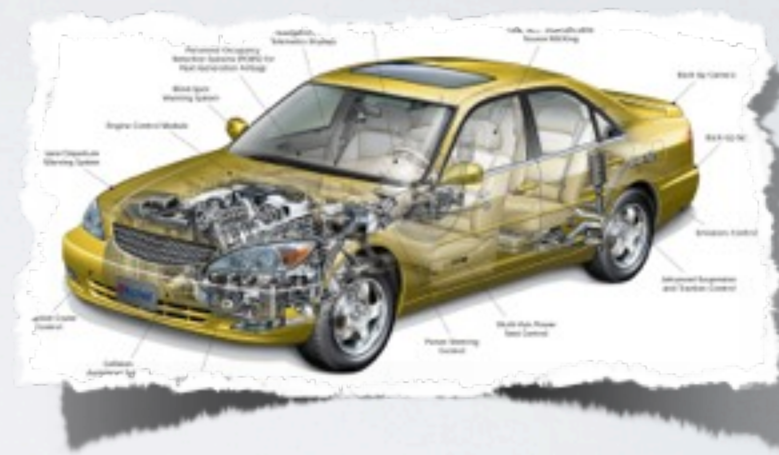
IMPOSSIBLE?

- Societies in nature

IMPOSSIBLE?

- Societies in nature

- Societies of artificial systems
  - A modern car is much more of a robot than a simple machine!



IMPOSSIBLE?

- Societies in nature

- Societies of artificial systems

- Streets with cars following driving rules are indeed SoR:

IMPOSSIBLE?

- Societies in nature

- Societies of artificial systems

- Streets with cars following driving rules are indeed SoR:

  **Many, *Fast/Slow, *Different,*
  **Autonomous, *Uncoordinated,*
  **Competing, *Joining in and quitting at will,*
  **Mostly well-behaved ...except for few jerks*

WHY DO WE NEED A SOR?

- From industrial robots to service robots, ... and finally to *robots everywhere*.

WHY DO WE NEED A SOR?

- From industrial robots to service robots, ... and finally to *robots everywhere*.

WHY DO WE NEED A SOR?

- From industrial robots to service robots, ... and finally to *robots everywhere*.

- low reconfigurability (e.g. the end-effector can be changed)

WHY DO WE NEED A SOR?

- From industrial robots to service robots, ... and finally to *robots everywhere*.

- low reconfigurability (e.g. the end-effector can be changed)
- almost completely specified operating conditions (closed and obstacle-free spaces)
- limited Robot-Robot Interaction (ad-hoc solutions with few identical copies of robots)

WHY DO WE NEED A SOR?

- From industrial robots to service robots, ... and finally to *robots everywhere*.

- low reconfigurability (e.g. the end-effector can be changed)
- almost completely specified operating conditions (closed and obstacle-free spaces)
- limited Robot-Robot Interaction (ad-hoc solutions with few identical copies of robots)

- *A robot must be rethought of as a physical entity embedded in a full-fledge society, interacting through social rules, not jeopardizing others' efforts.*

MANY CHALLENGES

- How do we specify social robots' behavior?
  - *Models / Formal languages*

MANY CHALLENGES

- How do we specify social robots' behavior?
  - *Models / Formal languages*
- Localism implies partial/uncertain knowledge -> How do we reach consistent social behavior?
  - *Agreement mechanisms for global view/information reconstruction*

MANY CHALLENGES

- How do we specify social robots' behavior?
    - *Models / Formal languages*
- Localism implies partial/uncertain knowledge -> How do we reach consistent social behavior?
    - *Agreement mechanisms for global view/information reconstruction*
- Openness implies partial behavior knowledge -> How can robots learn others' dynamics and social rules?
    - *Species classification [Martini, Egerstedt, Bicchi]*
    - *Logical function learning by observations*

MANY CHALLENGES

- How do we specify social robots' behavior?
    - *Models / Formal languages*
- Localism implies partial/uncertain knowledge -> How do we reach consistent social behavior?
    - *Agreement mechanisms for global view/information reconstruction*
- Openness implies partial behavior knowledge -> How can robots learn others' dynamics and social rules?
    - *Species classification [Martini, Egerstedt, Bicchi]*
    - *Logical function learning by observations*
- How do robots authenticate and establish trust relations? [Dini]

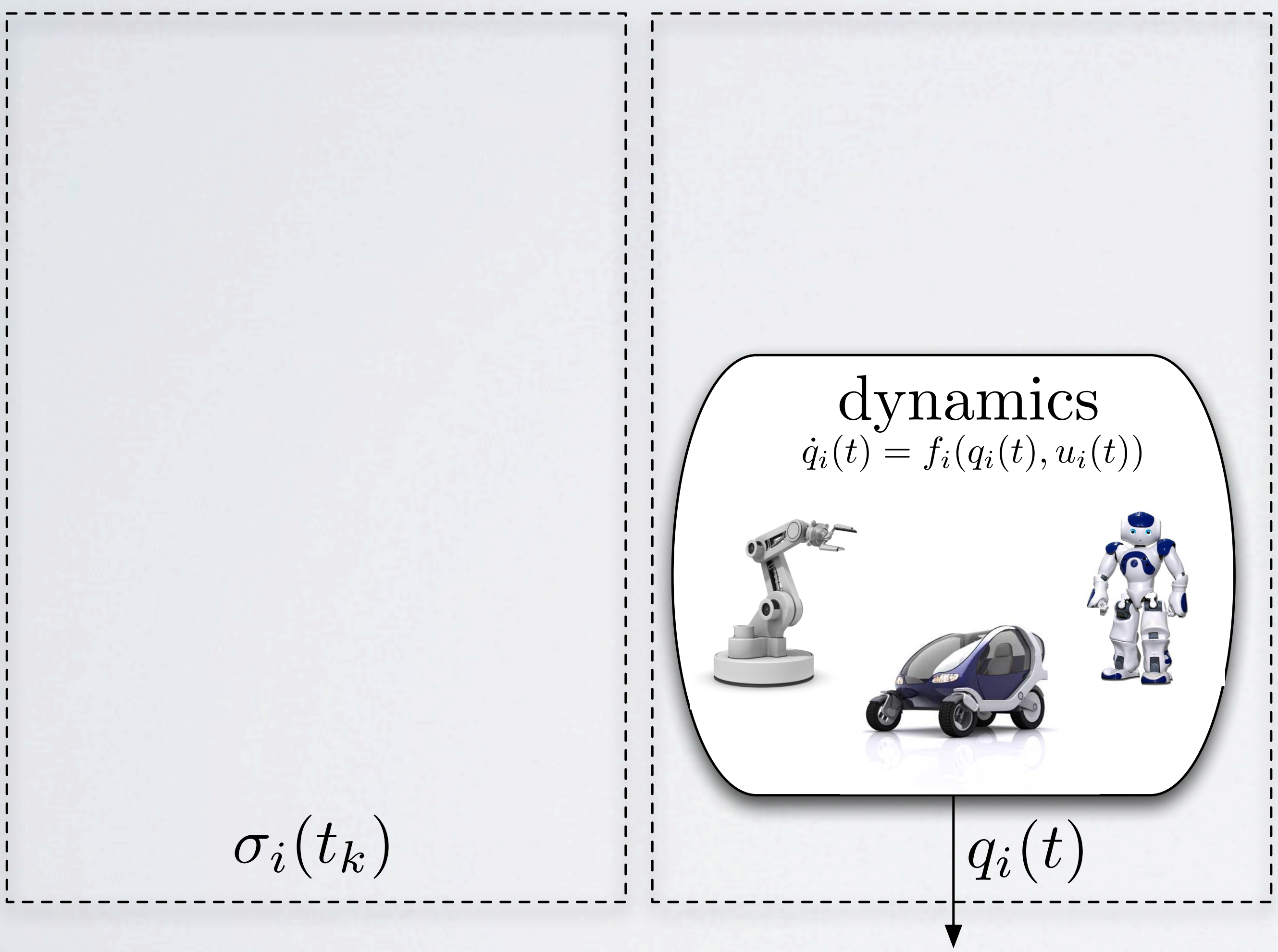
SOCIAL ROBOTS' MODELING

- a social robot is a hybrid system

$\sigma_i(t_k)$ $q_i(t)$

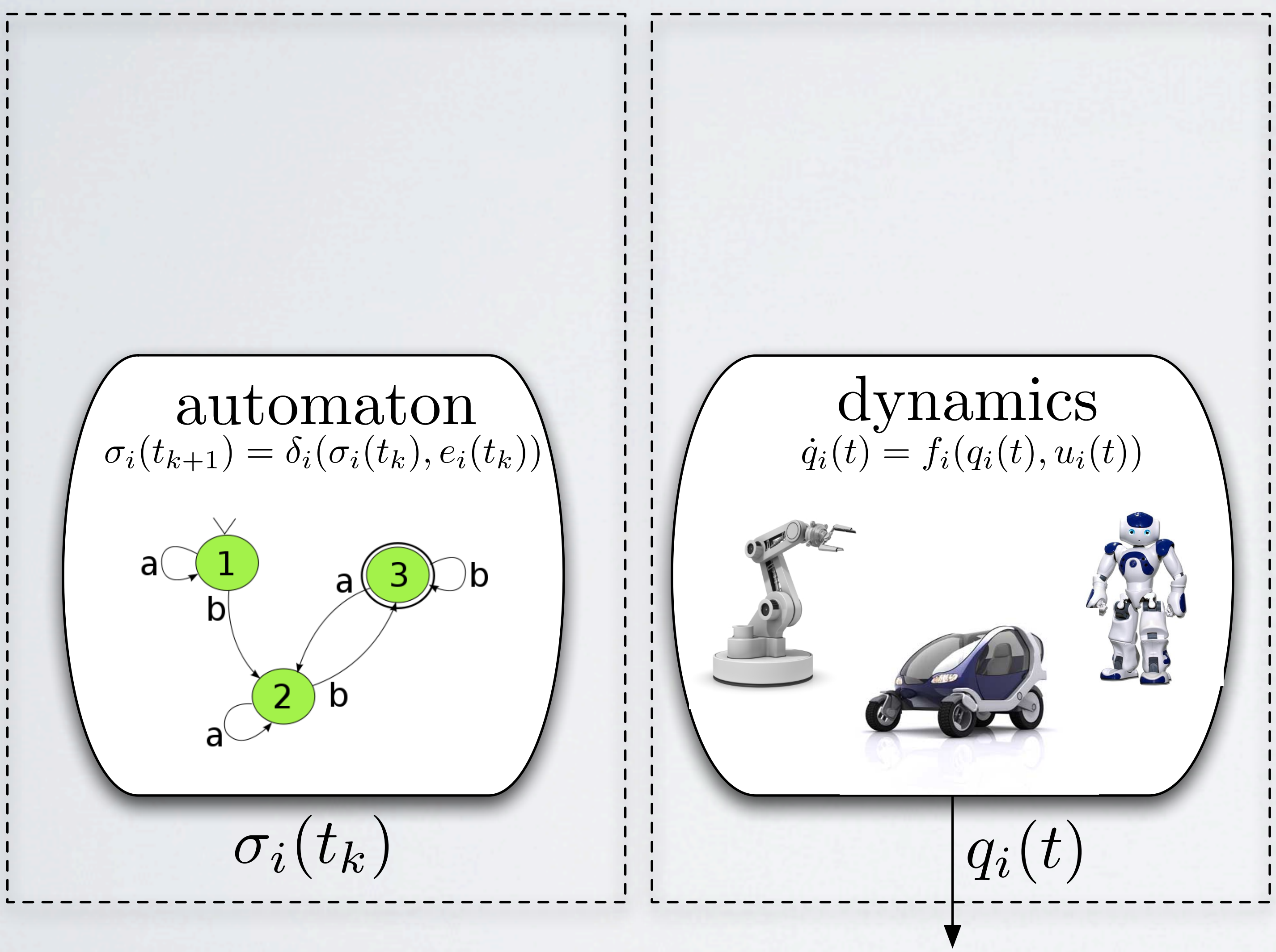
SOCIAL ROBOTS' MODELING

- a social robot is a hybrid system



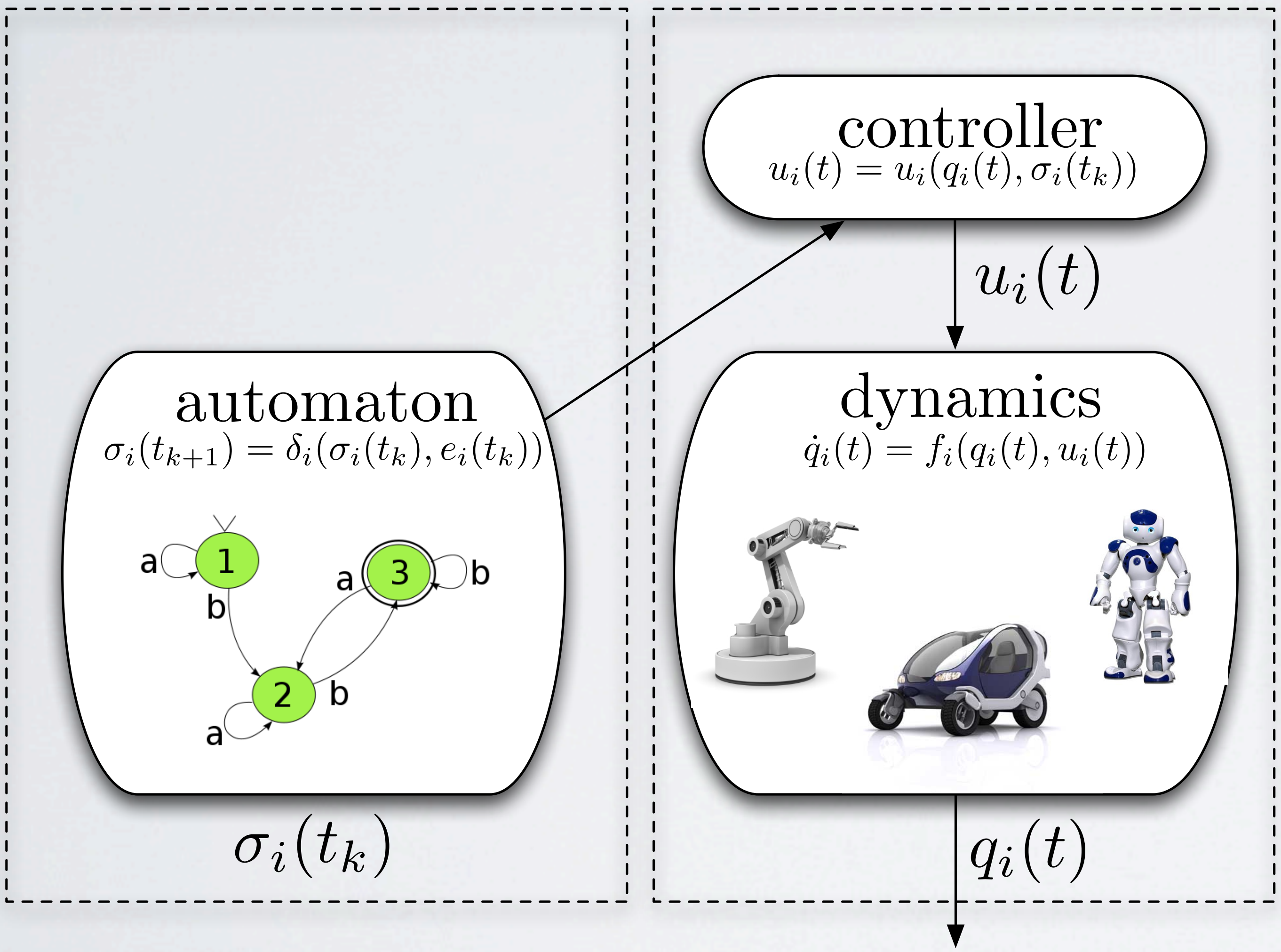
SOCIAL ROBOTS' MODELING

- a social robot is a hybrid system



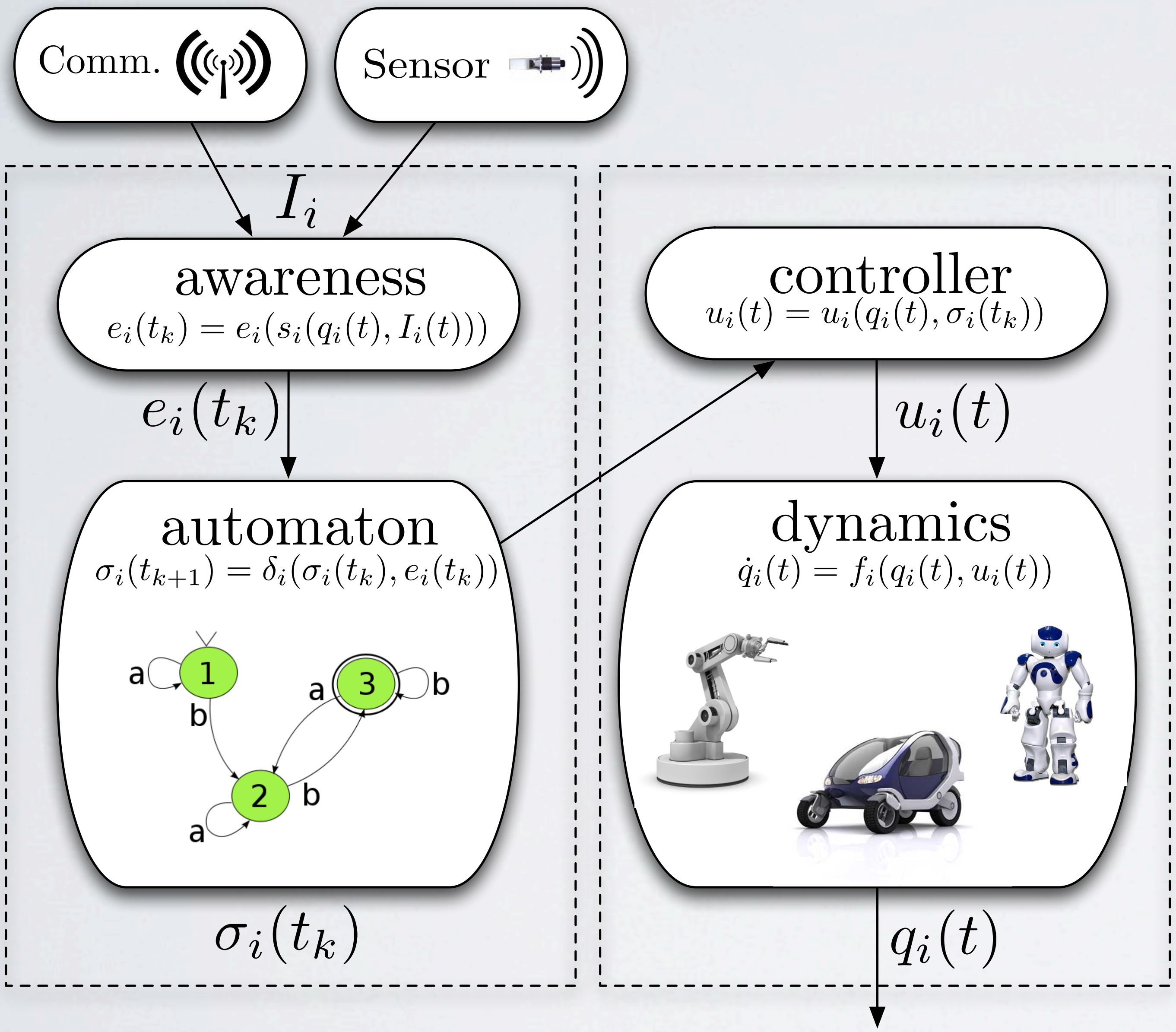
SOCIAL ROBOTS' MODELING

- a social robot is a hybrid system



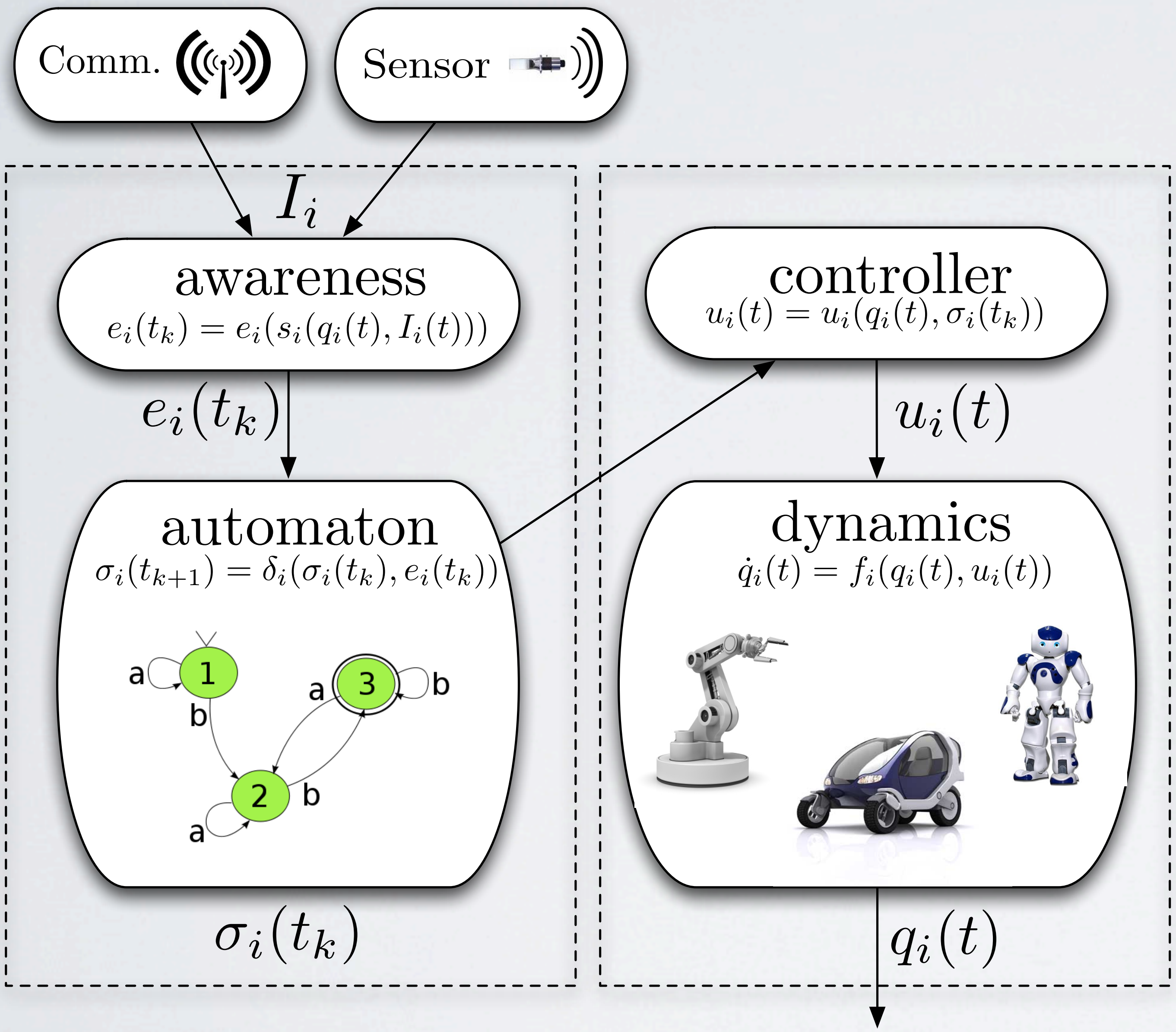
SOCIAL ROBOTS' MODELING

- a social robot is a hybrid system



SOCIAL ROBOTS' MODELING

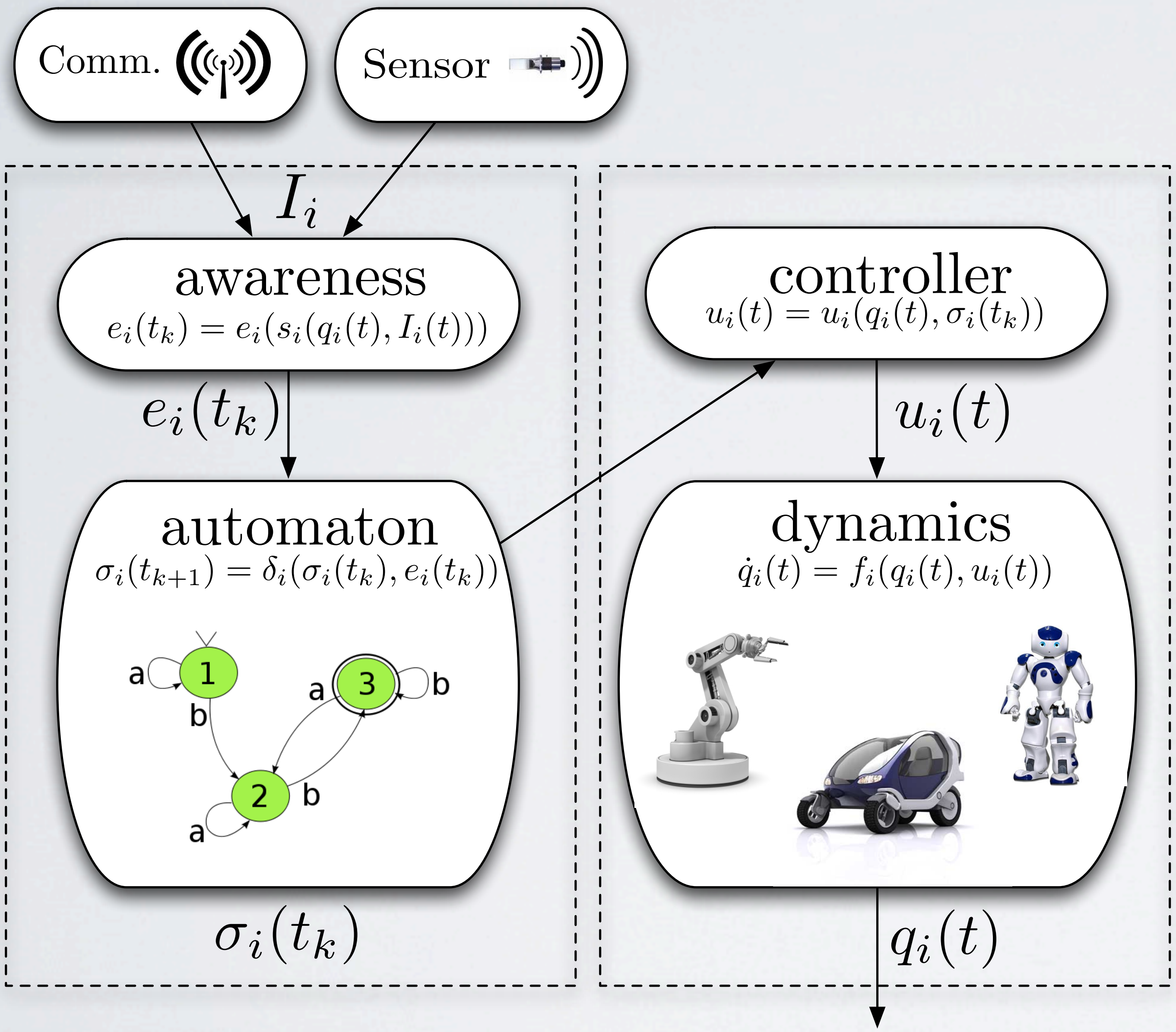
- a social robot is a hybrid system



- natural-like specification language

SOCIAL ROBOTS' MODELING

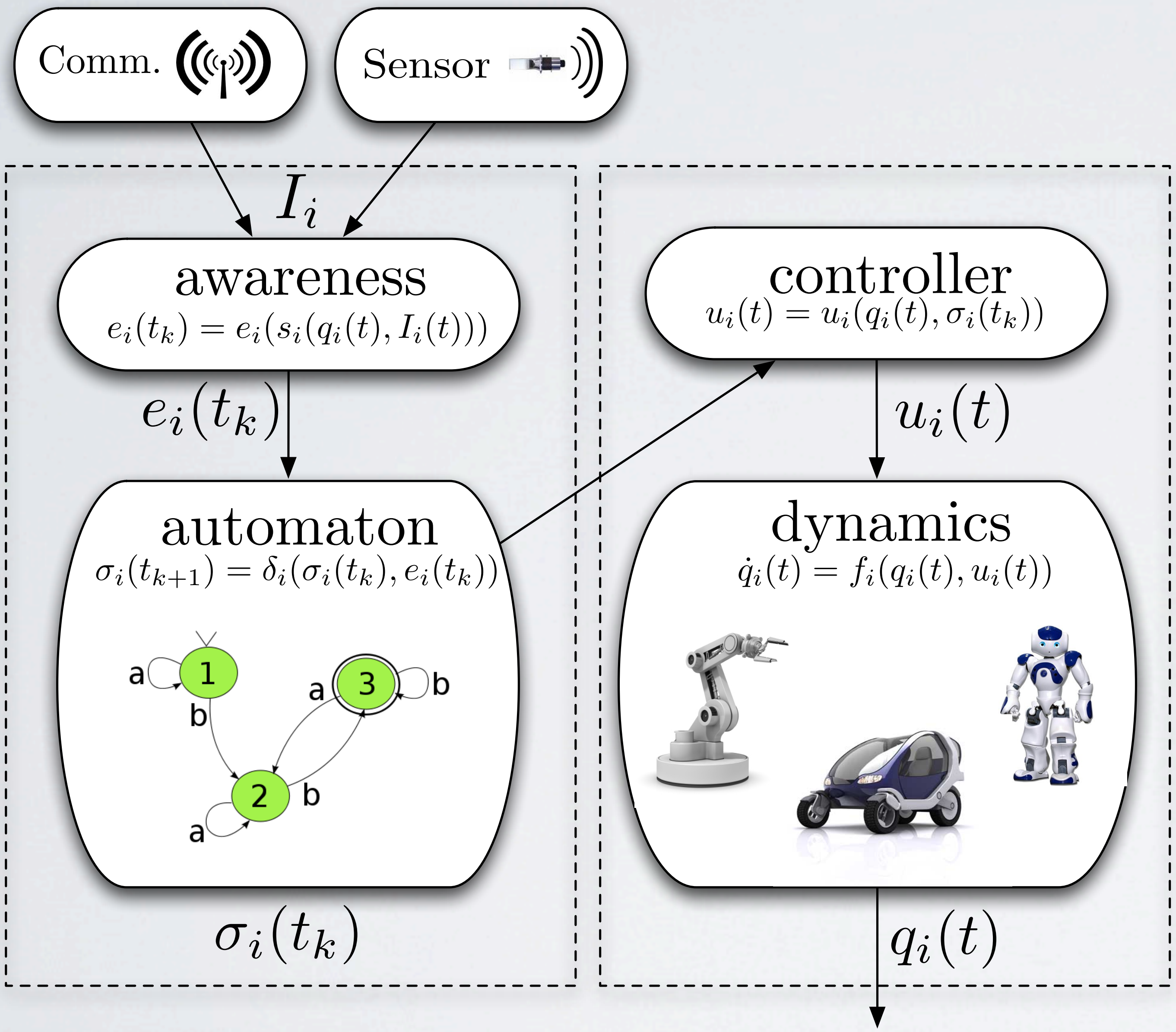
- a social robot is a hybrid system



- natural-like specification language
- makers are only required to adhere to standard spec. protocols

SOCIAL ROBOTS' MODELING

- a social robot is a hybrid system



- natural-like specification language
- makers are only required to adhere to standard spec. protocols
- an entire SoR can be described as a computer program:
  - complex behaviors by few keywords and a grammar
  - composability, reusability, ...

SOCIETY AGREEMENT

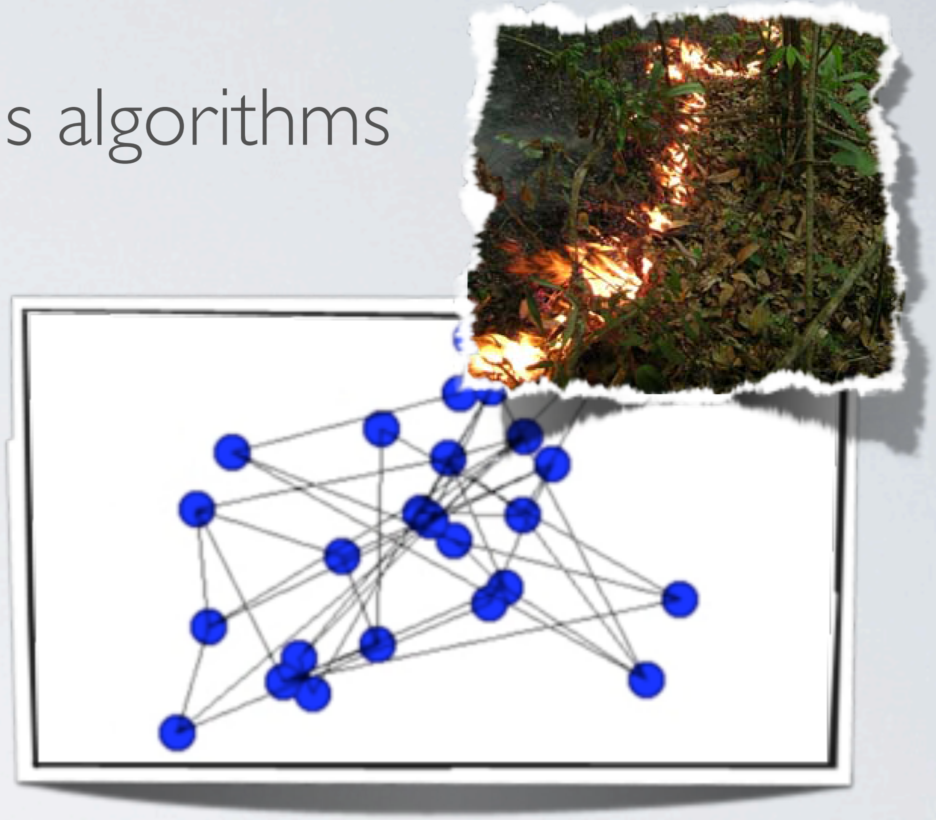
- Global information reconstruction by consensus algorithms

SOCIETY AGREEMENT

- Global information reconstruction by consensus algorithms

- In the easiest case, the quantity to agree about is real and a *linear iterative strategy* can be used to update a robot's state:

$$x_i(t+1) = \sum_{j=0}^{n} a_{i,j}(x_j(t) - x_i(t))$$

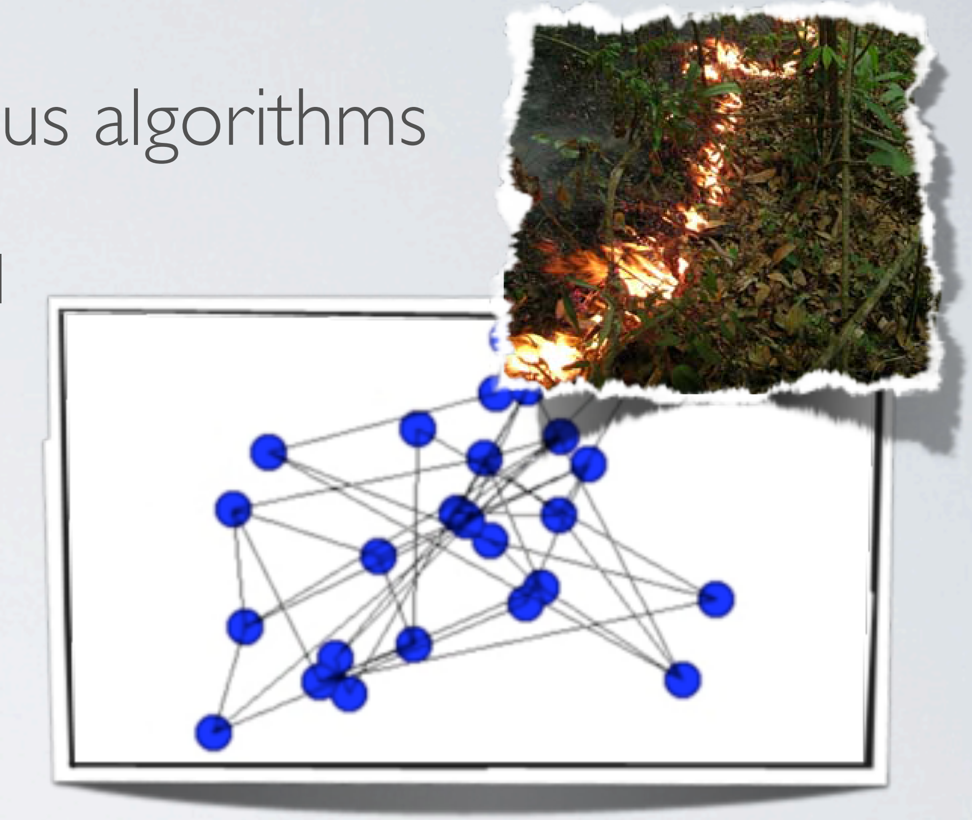


SOCIETY AGREEMENT

- Global information reconstruction by consensus algorithms

- In the easiest case, the quantity to agree about is real and a *linear iterative strategy* can be used to update a robot's state:

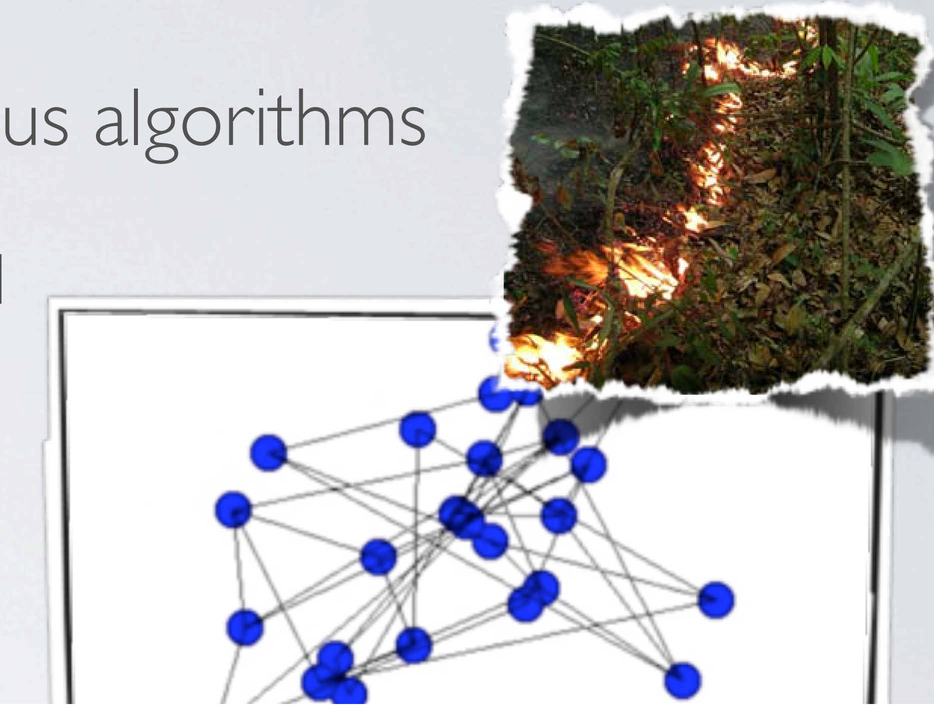
$$x_i(t+1) = \sum_{j=0}^{n} a_{i,j}(x_j(t) - x_i(t))$$



- Optimal sensor coverage by *nonlinear iterative strategy* based on Voronoi tessellation

$$\dot{q}_i = -k(q_i - C_{\mathcal{V}_i}(q_i))$$

SOCIETY AGREEMENT

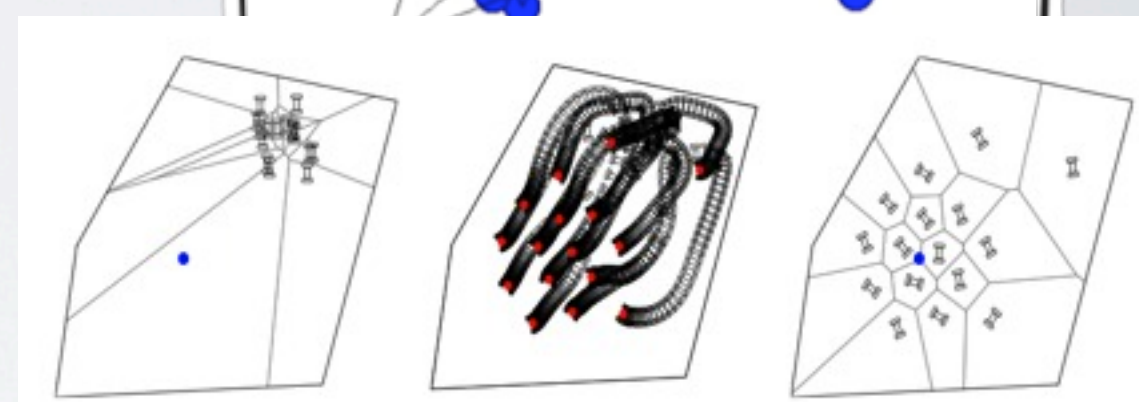


- Global information reconstruction by consensus algorithms
  - In the easiest case, the quantity to agree about is real and a *linear iterative strategy* can be used to update a robot's state:

$$x_i(t+1) = \sum_{j=0}^{n} a_{i,j}(x_j(t) - x_i(t))$$

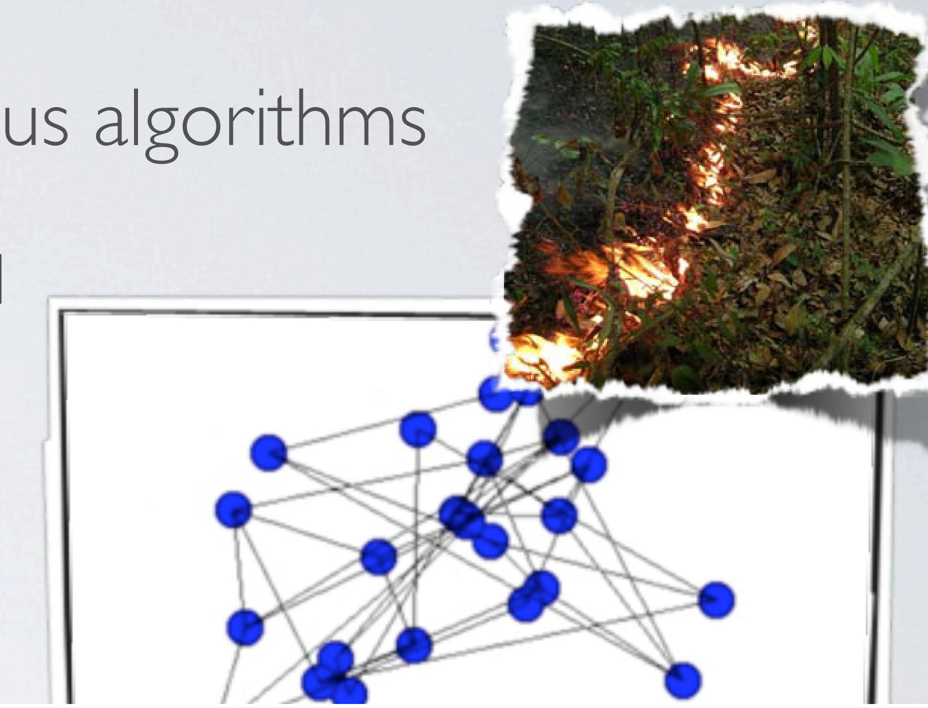
  - Optimal sensor coverage by *nonlinear iterative strategy* based on Voronoi tessellation

$$\dot{q}_i = -k(q_i - C_{\mathcal{V}_i}(q_i))$$



SOCIETY AGREEMENT

- Global information reconstruction by consensus algorithms

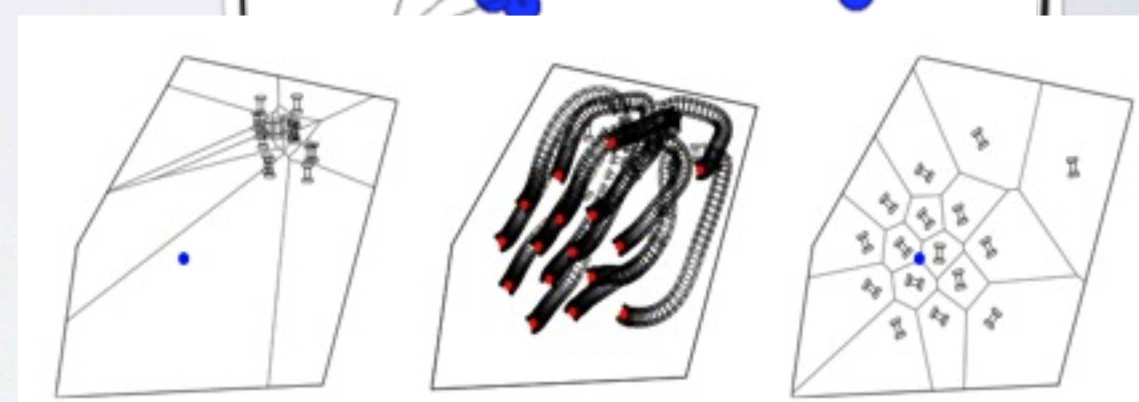


- In the easiest case, the quantity to agree about is real and a *linear iterative strategy* can be used to update a robot's state:

$$x_i(t+1) = \sum_{j=0}^{n} a_{i,j}(x_j(t) - x_i(t))$$

- Optimal sensor coverage by *nonlinear iterative strategy* based on Voronoi tessellation

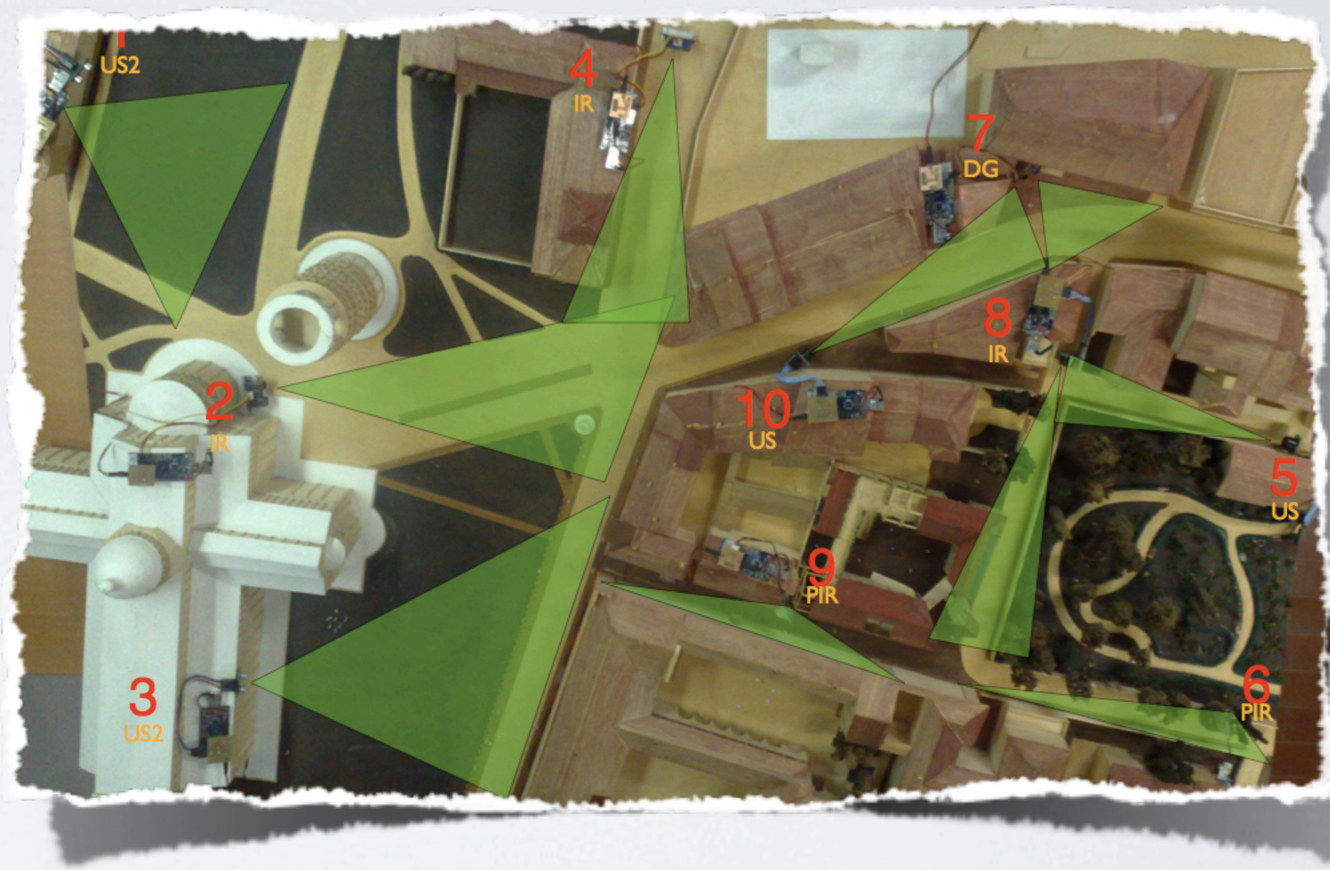
$$\dot{q}_i = -k(q_i - C_{\mathcal{V}_i}(q_i))$$



- *Nonlinear set-valued iterative strategies* can solve other important problems:
  - Clock synchronization from confidence intervals [Marzullo, CASE'09],
  - Geographical chart reconstruction from partial/uncertain aerial snapshots, ...

INTRUSION / ATTACK DETECTION

- *Effective intrusion reaction requires global, consistent reconstruction of suspicious people / objects presence.*

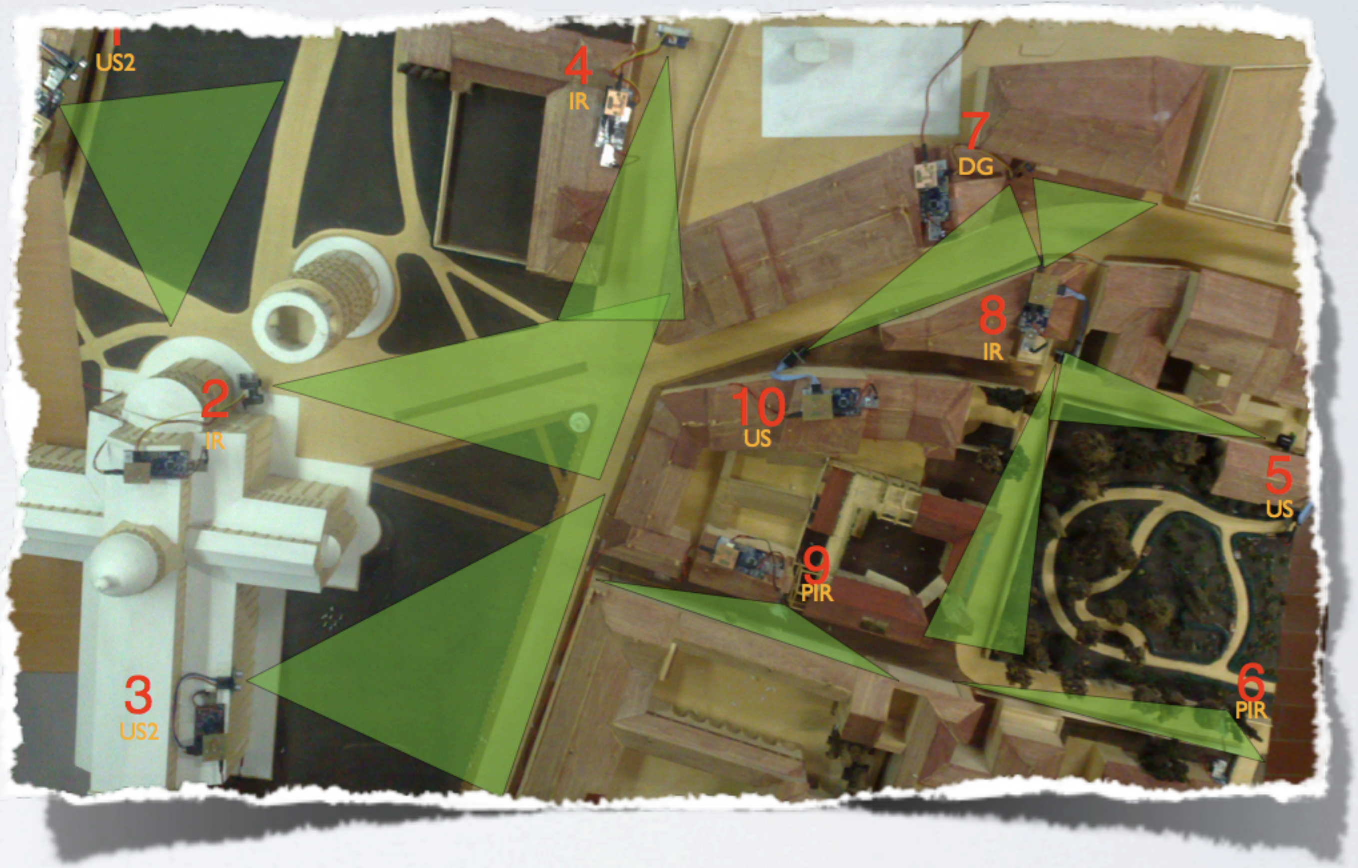


INTRUSION / ATTACK DETECTION

- *Effective intrusion reaction requires global, consistent reconstruction of suspicious people / objects presence.*

- a complex environment

- binary input events
  $$u_j \in \mathbb{B}$$



INTRUSION / ATTACK DETECTION

- *Effective intrusion reaction requires global, consistent reconstruction of suspicious people / objects presence.*

- a complex environment

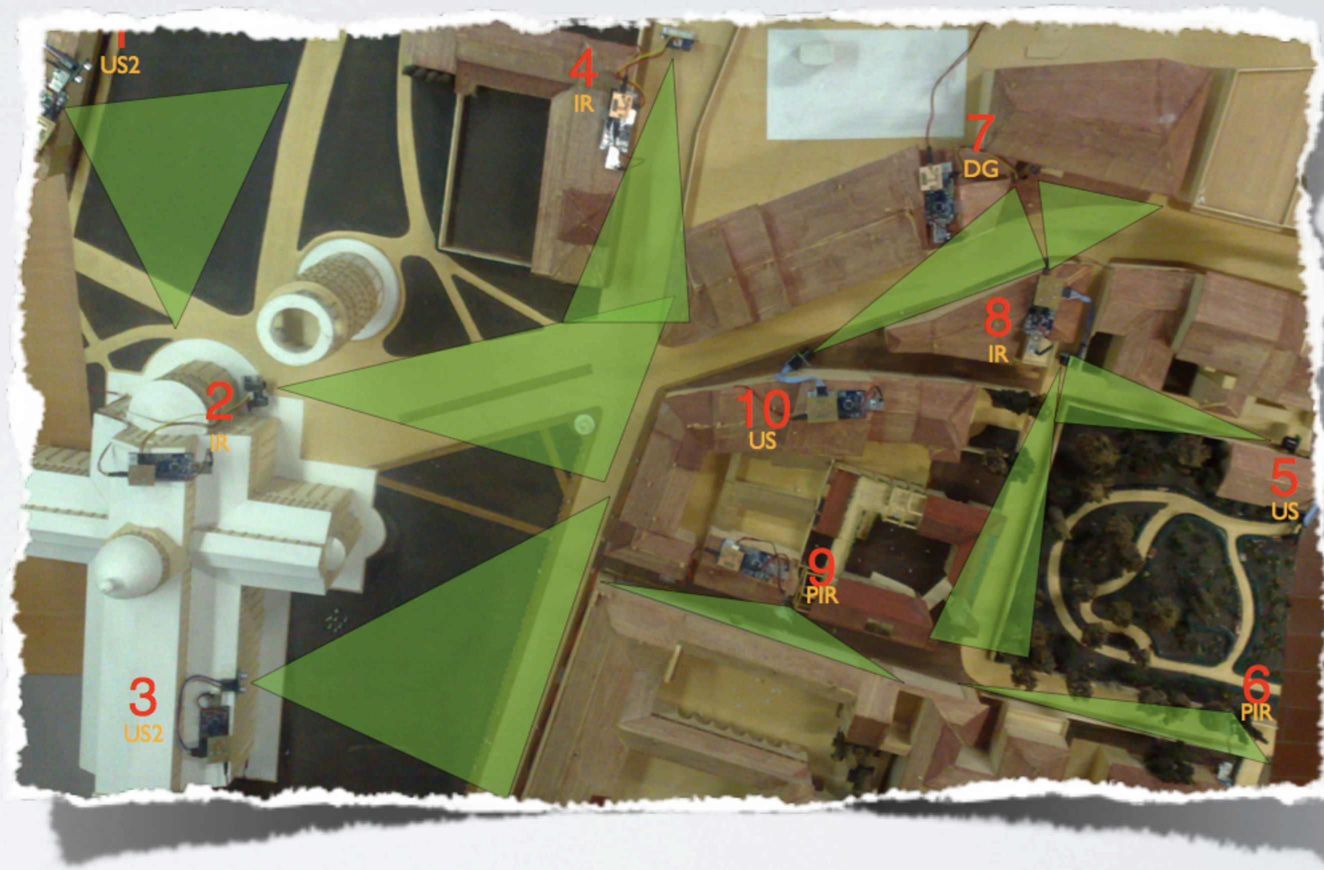
- binary input events

$$u_j \in \mathbb{B}$$

- guards with limited visibility and communication

- binary alarm states

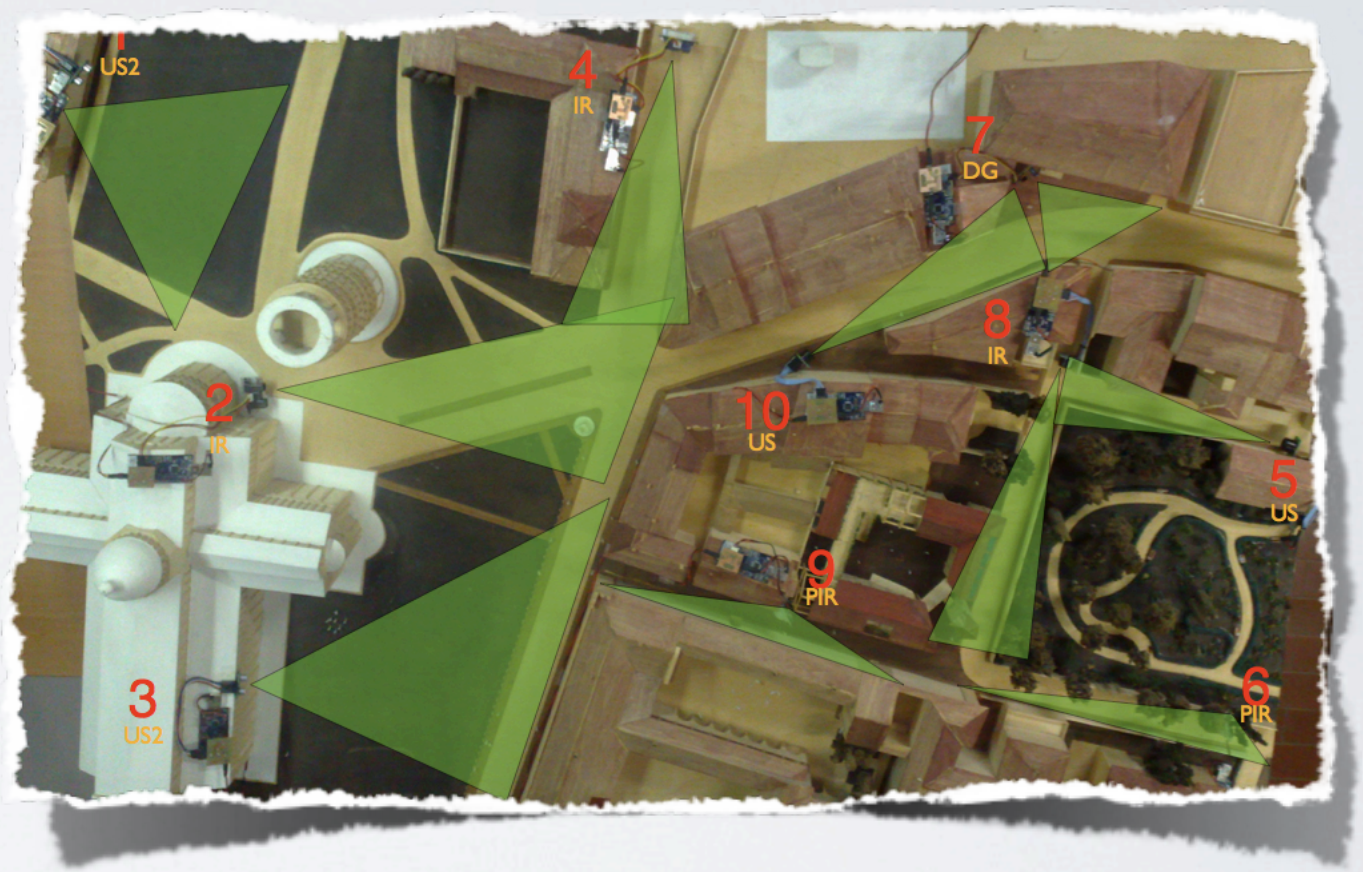
$$X_i = (0, 1, 1, 0, 0, 1, 0, 0, 0, 1)$$



INTRUSION / ATTACK DETECTION

- *Effective intrusion reaction requires global, consistent reconstruction of suspicious people / objects presence.*

- a complex environment



- binary input events

$$u_j \in \mathbb{B}$$

- guards with limited visibility and communication

- binary alarm states

$$X_i = (0, 1, 1, 0, 0, 1, 0, 0, 0, 1)$$

- *only neighbor-to-neighbor interaction is possible!*

A GENERAL SHARED FRAMEWORK

- (centralized) decisions depending on binary input events

$$\begin{cases} y_1 = f_1(u_1, \ldots, u_m)\,, \\ \cdots \\ y_p = f_p(u_1, \ldots, u_m)\,, \end{cases}$$

$$u_j, y_i \in \mathbb{B} = \{0, 1\}$$

A GENERAL SHARED FRAMEWORK

- (centralized) decisions depending on binary input events

$$\begin{cases} y_1 = f_1(u_1, \ldots, u_m)\,, \\ \ldots \\ y_p = f_p(u_1, \ldots, u_m)\,, \end{cases}$$

$$u_j, y_i \in \mathbb{B} = \{0, 1\}$$

- partial input visibility

$$V_{i,j} = \begin{cases} 1 & \text{if agent } i \text{ ``sees'' } u_j \\ 0 & \text{otherwise} \end{cases}$$

- limited communication

$$C_{i,j} = \begin{cases} 1 & i \text{ can receive a message from } j \\ 0 & \text{otherwise} \end{cases}$$

- a binary vector state

$$X_i = (x_{i,1}, \cdots, x_{i,p})$$

A GENERAL SHARED FRAMEWORK

- (centralized) decisions depending on binary input events

$$\begin{cases} y_1 = f_1(u_1, \ldots, u_m)\,, \\ \cdots \\ y_p = f_p(u_1, \ldots, u_m)\,, \end{cases}$$

$$u_j, y_i \in \mathbb{B} = \{0, 1\}$$

- partial input visibility

$$V_{i,j} = \begin{cases} 1 & \text{if agent } i \text{ ``sees'' } u_j \\ 0 & \text{otherwise} \end{cases}$$

- limited communication

$$C_{i,j} = \begin{cases} 1 & i \text{ can receive a message from } j \\ 0 & \text{otherwise} \end{cases}$$

- a binary vector state

$$X_i = (x_{i,1}, \cdots, x_{i,p})$$

Logical Consensus Problem: Given visibility and adjacency matrices, $V = \{V_{i,j}\}$ and $C = \{C_{i,j}\}$, design a distributed protocol of the form

$$X(t+1) = F(X(t), u(t))$$

converging to the consensus state $\mathbf{1}_n f(u_1, \cdots, u_m)$, from any $X(0)$.

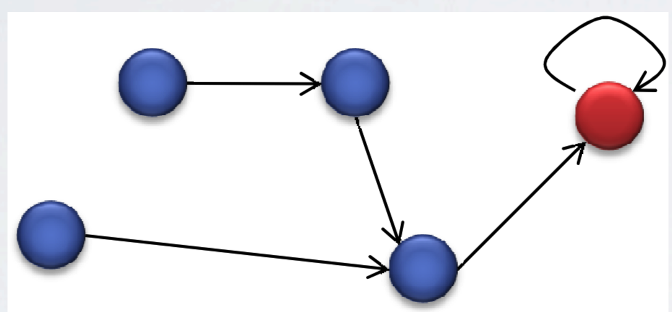
LOGICAL ITERATIONS

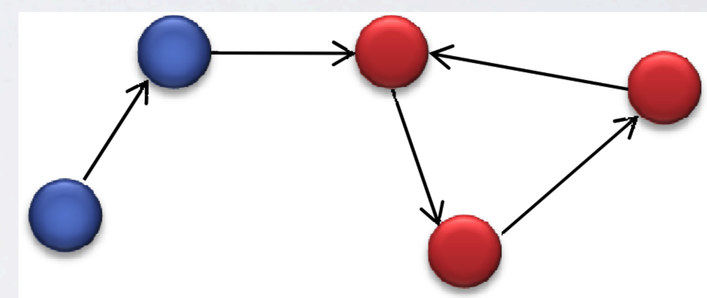
- Traditional approaches do not apply to logical maps $F : \mathbb{B}^n \times \mathbb{B}^m \to \mathbb{B}^n$, but tools for the convergence of automata are of use [Robert'74-'83].

LOGICAL ITERATIONS

- Traditional approaches do not apply to logical maps $F : \mathbb{B}^n \times \mathbb{B}^m \to \mathbb{B}^n$, but tools for the convergence of automata are of use [Robert'74-'83].

- Iterations over finite domains either converge to an equilibrium or to a cycle

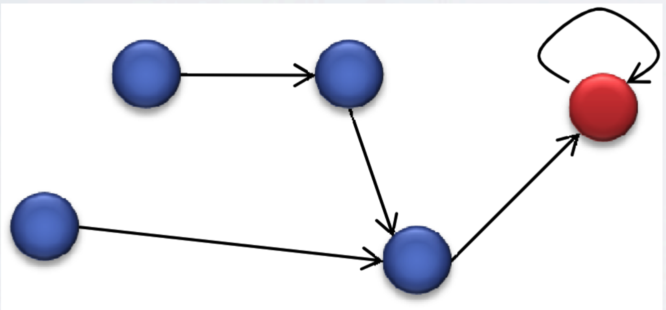


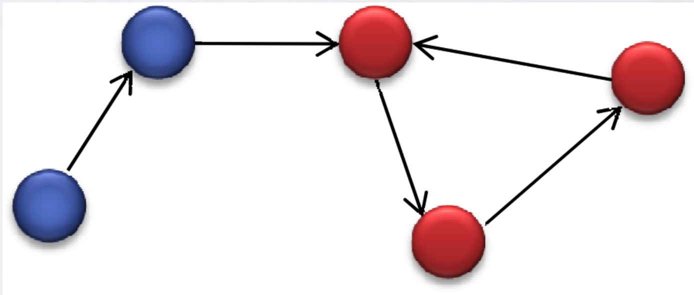


LOGICAL ITERATIONS

- Traditional approaches do not apply to logical maps $F : \mathbb{B}^n \times \mathbb{B}^m \rightarrow \mathbb{B}^n$, but tools for the convergence of automata are of use [Robert'74-'83].

- Iterations over finite domains either converge to an equilibrium or to a cycle





- the incidence matrix of a logical map F is a binary matrix $B(F) = \{b_{i,j}\}$
  ($b_{i,j} = 1$ if the j-th element of $F(x)$ depends on the i-th element of x)

$$F(x) = \begin{pmatrix} x_1 + x_2 + x_3 \\ \neg x_3 \\ 1 \end{pmatrix} \rightarrow B(F) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

LOGICAL ITERATIONS - GLOBAL CONVERGENCE

- the eigenvalues of a binary matrix A are all the scalars

$$\lambda \in \mathbb{B} \text{ s.t. } \exists\, x \in \mathbb{B}^n \text{ s.t. } A\,x = \lambda\,x$$

LOGICAL ITERATIONS - GLOBAL CONVERGENCE

- the eigenvalues of a binary matrix A are all the scalars

$$\lambda \in \mathbb{B} \text{ s.t. } \exists\, x \in \mathbb{B}^n \text{ s.t. } A\,x = \lambda\,x$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \Rightarrow$$

$$A_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$A_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

LOGICAL ITERATIONS - GLOBAL CONVERGENCE

- the eigenvalues of a binary matrix A are all the scalars

$$\lambda \in \mathbb{B} \text{ s.t. } \exists x \in \mathbb{B}^n \text{ s.t. } A\,x = \lambda\,x$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$A_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$A_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

- its spectral radius $\rho(A)$ is the biggest eigenvalue

- $\rho(A) = 0 \quad \leftrightarrow \quad A$ is similar to a strictly lower/upper matrix

LOGICAL ITERATIONS - GLOBAL CONVERGENCE

- the eigenvalues of a binary matrix A are all the scalars

$$\lambda \in \mathbb{B} \text{ s.t. } \exists x \in \mathbb{B}^n \text{ s.t. } A\,x = \lambda\,x$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \quad A_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0 \qquad A_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

- its spectral radius $\rho(A)$ is the biggest eigenvalue

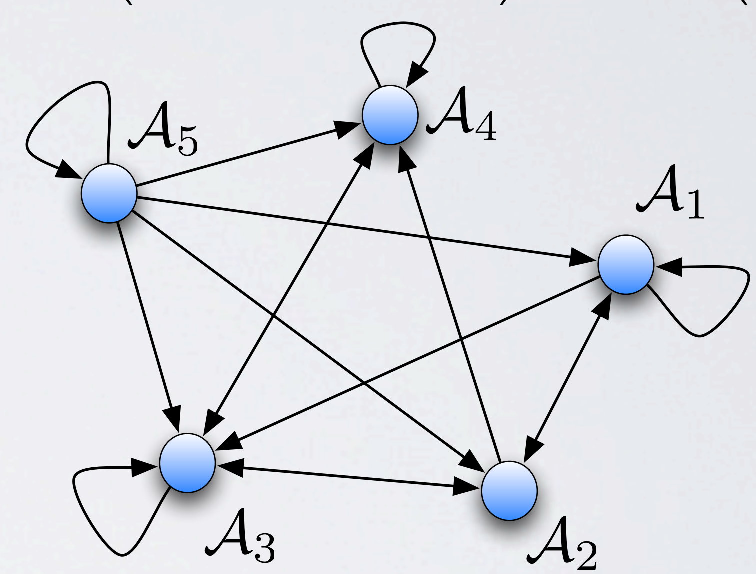
- $\rho(A) = 0 \quad \leftrightarrow \quad A$ is similar to a strictly lower/upper matrix

Theorem: A logical iterative system $x^+ = F(x)$ globally converges in finite time to its unique equilibrium if $\rho(B(F)) = 0$

WHEN IS CONSENSUS FEASIBLE?

- *Can every robot be reached by every input?*

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; V_j = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

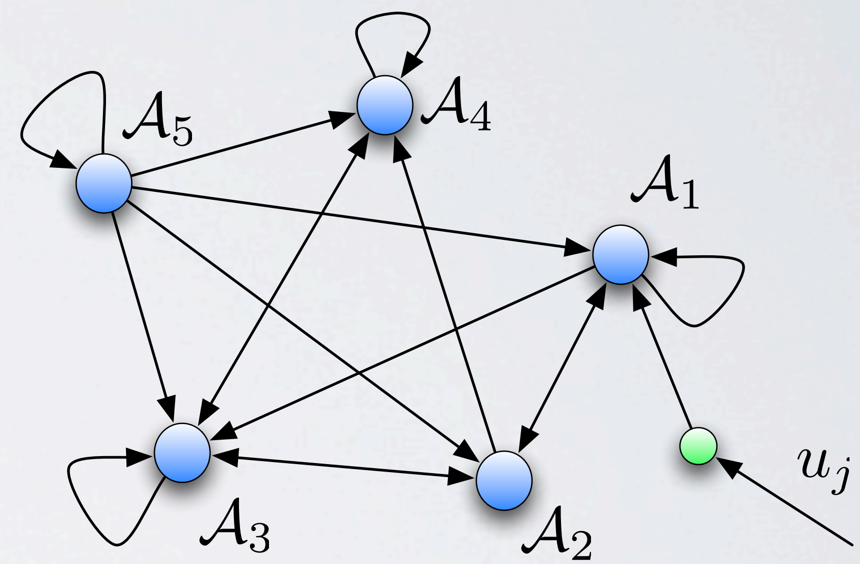


WHEN IS CONSENSUS FEASIBLE?

- *Can every robot be reached by every input?*

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; V_j = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

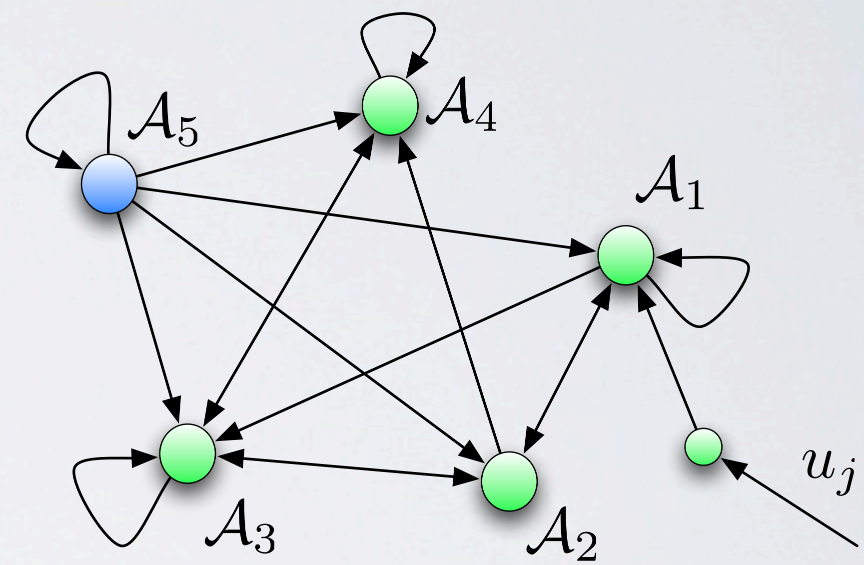
- augment the graph with a node for each input
- determine the existence of paths from each input node to every robot



WHEN IS CONSENSUS FEASIBLE?

- *Can every robot be reached by every input?*
  - augment the graph with a node for each input
  - determine the existence of paths from each input node to every robot
  - introducing a *binary reachability matrix*

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; V_j = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



$$R_j = \left(V_j \; CV_j \; C^2 V_j \; \cdots \; C^{n-1} V_j \right)$$

$$\mathrm{span}(R_j) = \{ i \,|\, I_j(n-1) = 1 \}$$

$$I_j = \sum_{k=0}^{n-1} C^k V_j = \sum_{k=0}^{n-1} R_j(:, i)$$

$$R_j = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to I_j(5) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathrm{span}(R_j) = \{1, 2, 3, 4\}$$

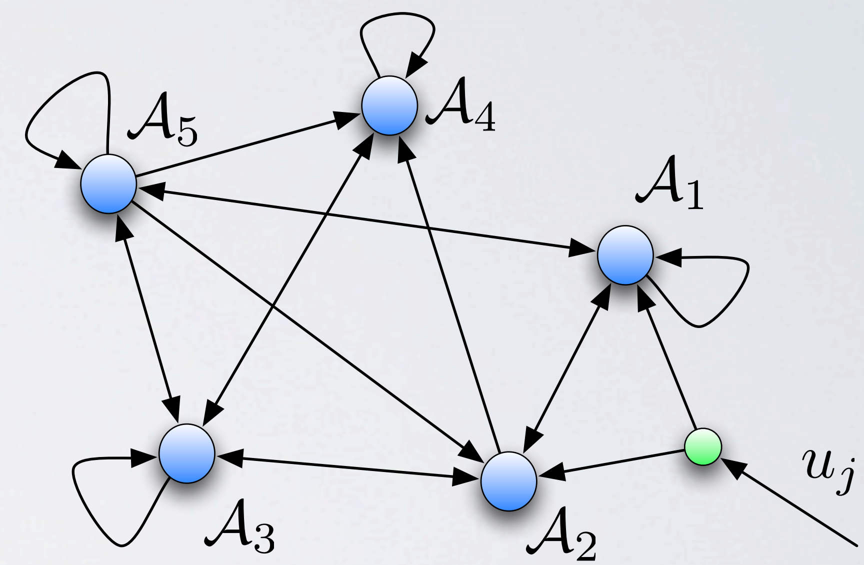
consensus is unfeasible!

DESIGN OF LOGICAL LINEAR CONSENSUS

- *What is the best way to propagate every input?*

Modified Example

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad V_j = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

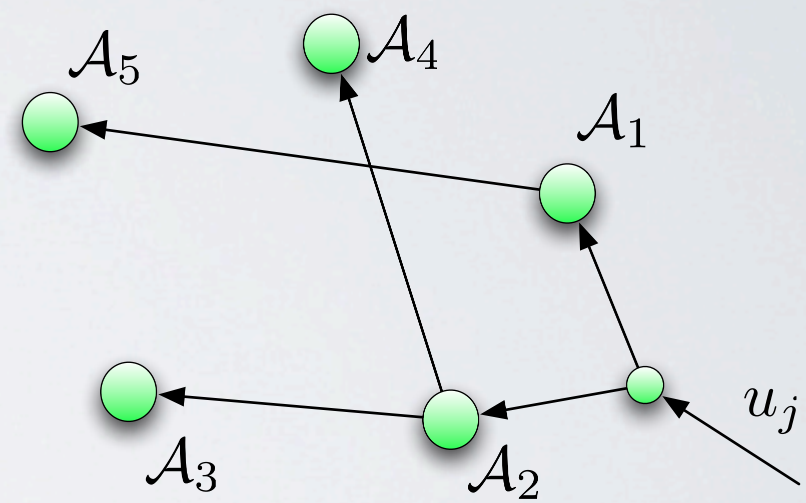


DESIGN OF LOGICAL LINEAR CONSENSUS

- *What is the best way to propagate every input?*

Modified Example

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad V_j = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

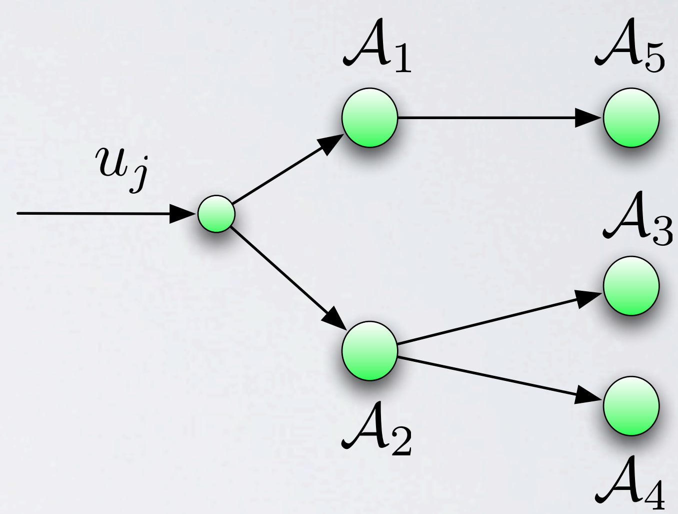


DESIGN OF LOGICAL LINEAR CONSENSUS

- *What is the best way to propagate every input?*

Modified Example

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad V_j = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

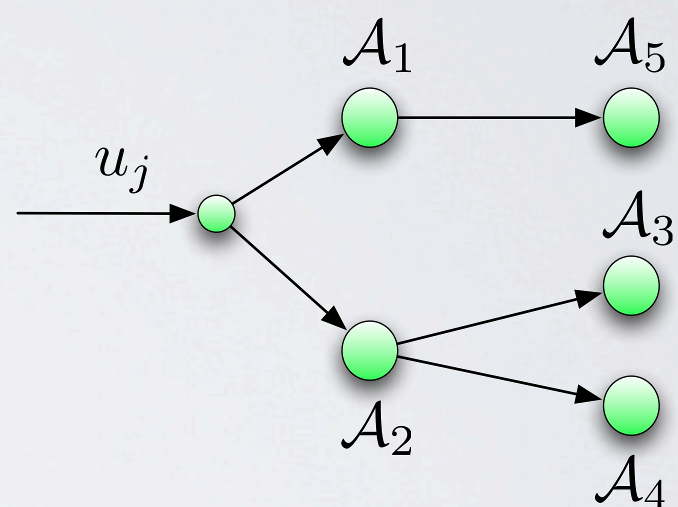


DESIGN OF LOGICAL LINEAR CONSENSUS

- *What is the best way to propagate every input?*

Modified Example

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad V_j = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



$$\begin{cases} x_{1,j}(t+1) = u_j(t) \\ x_{2,j}(t+1) = u_j(t) \\ x_{3,j}(t+1) = x_{2,j}(t) \\ x_{4,j}(t+1) = x_{2,j}(t) \\ x_{5,j}(t+1) = x_{1,j}(t) \end{cases}$$

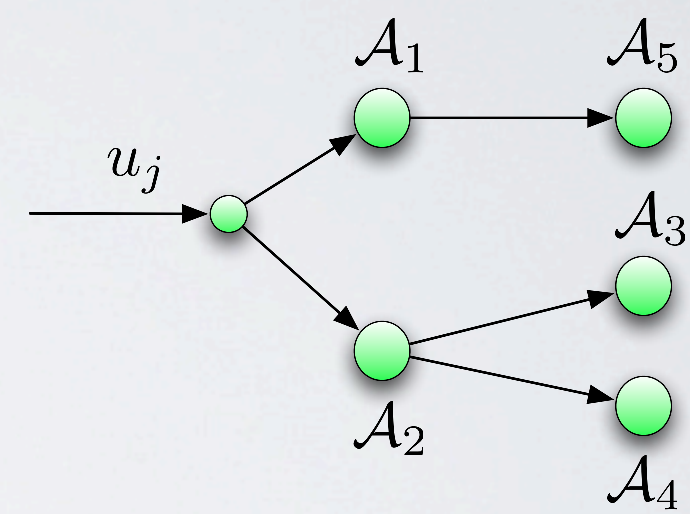
DESIGN OF LOGICAL LINEAR CONSENSUS

- *What is the best way to propagate every input?*

Modified Example

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad V_j = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- Let Cj* be the adjacency matrix of the j-th Input Propagation Spanning Tree (IPST)



Theorem: A logical linear iterative system

$$x^+ = C_j^* x + V_j u_j$$

- is (C, V_j) compliant $(B(F_j(x, u_j)) \leq (C|V_j))$,
- globally converges to the consensus equilibrium $\mathbf{1}_n u_j$,
- is optimal in terms of time and messages exchanged.

Proof: F_j is globally convergent as $\rho(B(F_j)) = 0$, and $\mathbf{1}_n u_j$ is an equilibrium by construction.

$$\begin{cases} x_{1,j}(t+1) = u_j(t) \\ x_{2,j}(t+1) = u_j(t) \\ x_{3,j}(t+1) = x_{2,j}(t) \\ x_{4,j}(t+1) = x_{2,j}(t) \\ x_{5,j}(t+1) = x_{1,j}(t) \end{cases}$$

CONSENSUS WITH ROBOT FAILURE

- spontaneous malfunctioning
- robots' selfishness (*unfairly trying to gain resource access*)
- malicious reprogramming

CONSENSUS WITH ROBOT FAILURE

- spontaneous malfunctioning
- robots' selfishness (*unfairly trying to gain resource access*)
- malicious reprogramming

$$x_i(t+1) = F_i(x(t), u_j(t)) \oplus d_i$$

CONSENSUS WITH ROBOT FAILURE

- spontaneous malfunctioning
- robots' selfishness (*unfairly trying to gain resource access*)
- malicious reprogramming

- nominal update function

$$x_i(t+1) = F_i(x(t), u_j(t)) \oplus d_i$$

- disjunctive or
- binary disturbance

operating condition	d_i
correct agent	0
inverted agent	1
stuck on 0	$F_i(x, u_j)$
stuck on 1	$\neg F_i(x, u_j)$

CONSENSUS WITH ROBOT FAILURE

- spontaneous malfunctioning
- robots' selfishness (*unfairly trying to gain resource access*)
- malicious reprogramming

- nominal update function

$$x_i(t+1) = F_i(x(t), u_j(t)) \oplus d_i$$

- binary disturbance
- disjunctive or

operating condition	d_i
correct agent	0
inverted agent	1
stuck on 0	$F_i(x, u_j)$
stuck on 1	$\neg F_i(x, u_j)$

- linear consensus rule $F_i = C_j^*(i,:)\,x + V_j(i)\,u_j$ with e.g. $d_1 \neq 0$ reaches a state such as

$$\bar{x} = (u_j \oplus d_1, u_j, u_j \oplus d_1, u_j, u_j \oplus d_1)^T \neq \mathbf{1}_n u_j\ ,$$

ROBUST LOGICAL CONSENSUS

- bounded number of faults among every robot's neighbors (γ)

ROBUST LOGICAL CONSENSUS

- bounded number of faults among every robot's neighbors (γ)
- Byzantine behaviors can be tolerated with $2\gamma + 1$ neighbors

$$x_i(t+1) = \begin{cases} 0 & if \;\; card(S_0(t)) > card(S_1(t))\,, \\ 1 & if \;\; card(S_0(t)) < card(S_1(t))\,, \end{cases}$$

$$S_z(t) = \{h \,|\, C_{i,h} = 1,\, x_h(t) = z\}$$

ROBUST LOGICAL CONSENSUS

- bounded number of faults among every robot's neighbors (γ)
  - Byzantine behaviors can be tolerated with $2\gamma + 1$ neighbors

$$x_i(t+1) = \begin{cases} 0 & if \;\; card(S_0(t)) > card(S_1(t)) \, , \\ 1 & if \;\; card(S_0(t)) < card(S_1(t)) \, , \end{cases}$$

$$S_z(t) = \{h \,|\, C_{i,h} = 1, \, x_h(t) = z\}$$

- *a binary reachability matrix and IPST with multiplicity r*

$$R_j^r(C, V_j) = \left(I_j^{(1)} \, I_j^{(2)} \, \cdots \, I_j^{(n)} \right) \, ,$$

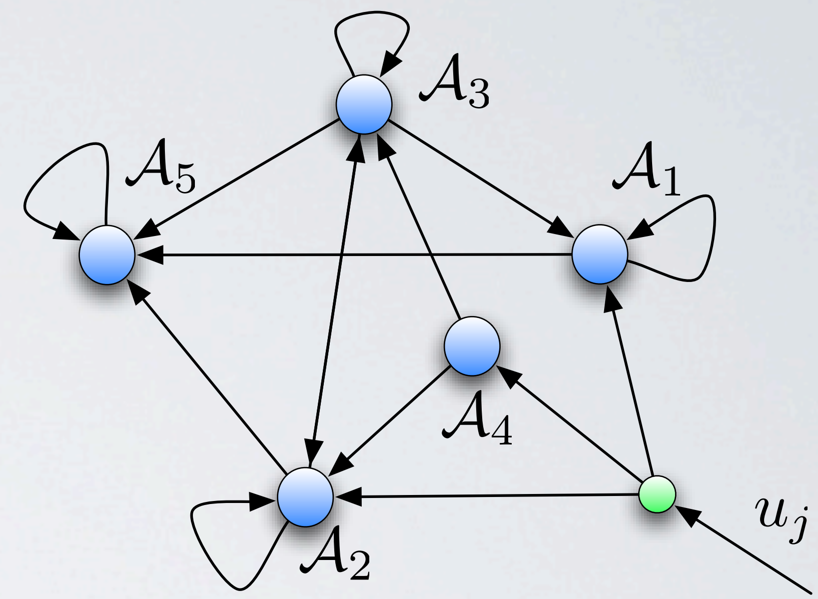
$$I_j^{(k)}(i) = \begin{cases} V_j(i) & k = 1 \, , \\ I_j^{(k-1)}(i) & k > 1, \, \mathrm{card}(K_i^k) < r \, , \\ 1 & k > 1, \, \mathrm{card}(K_i^k) \geq r \, , \end{cases}$$

$$K_i^k = \{h \,|\, C(i,h) I_j^{(k-1)}(h) = 1\}$$

ROBUST LOGICAL CONSENSUS: AN EXAMPLE

- Consider the following system with $\gamma = 1$ possible failures

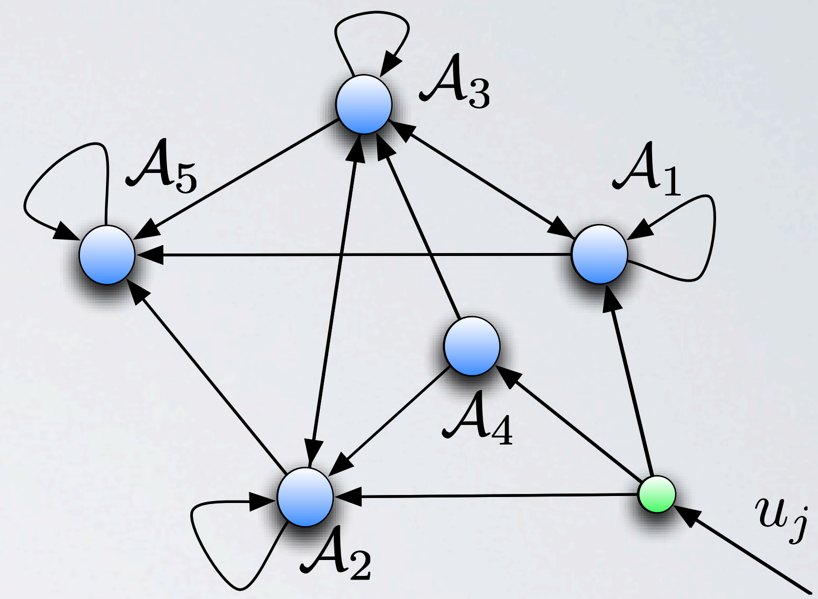
$$C = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \; V_j = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$



ROBUST LOGICAL CONSENSUS: AN EXAMPLE

- Consider the following system with $\gamma = 1$ possible failures

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \; V_j = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

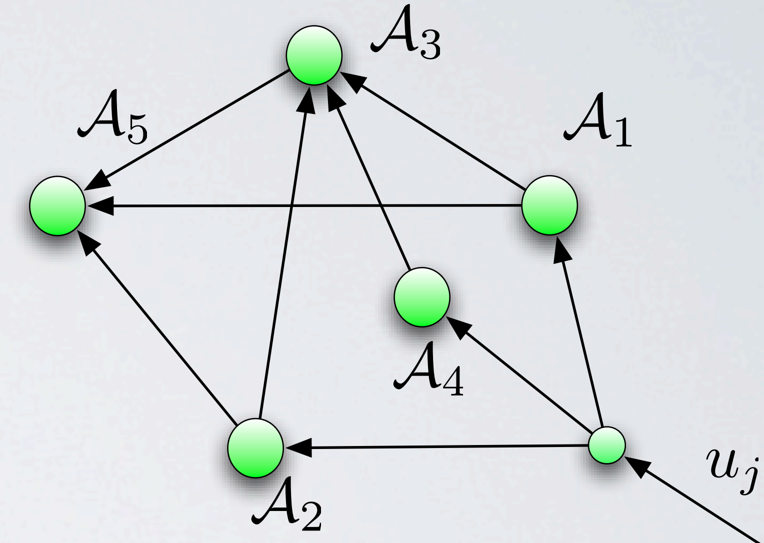


- *The pair (C,Vj) is 3-reachable!*

ROBUST LOGICAL CONSENSUS: AN EXAMPLE

- Consider the following system with $\gamma = 1$ possible failures

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \; V_j = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$



- *The pair (C,Vj) is 3-reachable!*

$$\begin{cases} x_1(t+1) = u_j(t)\,, \\ x_2(t+1) = u_j(t)\,, \\ x_3(t+1) = x_1(t)\,x_2(t) + x_1(t)\,x_4(t) + x_2(t)\,x_4(t)\,, \\ x_4(t+1) = u_j(t)\,, \\ x_5(t+1) = x_1(t)\,x_2(t) + x_1(t)\,x_3(t) + x_2(t)\,x_3(t)\,. \end{cases}$$

NONLINEAR LOGICAL CONSENSUS CONVERGENCE

Majority rule

NONLINEAR LOGICAL CONSENSUS CONVERGENCE

Theorem: Given a $(2\gamma+1)$–reachable pair (C, V_j) the nonlinear logical system

$$\begin{cases} x(t+1) &= F^*(x(t), u_j(t)), \\ x(0) &= x^0, \end{cases}$$

with

Majority rule

$$\begin{aligned} F_i^* \quad : \quad & \mathbb{B}^n \times \mathbb{B} \to \mathbb{B} \\ (x, u_j) \mapsto & \begin{cases} u_j & if \;\; V_j(i) = 1, \\ \sum_{H \in S_i^*} \Pi_{h \in H} x_h & if \;\; V_j(i) = 0, \end{cases} \end{aligned}$$

- is (C, V_j)–compliant, and
- globally converges in finite time to $\bar{x}_i = u_j$ for all correct i.

Proof: F_j is globally convergent as $\rho(B(F_j)) = 0$, and $\mathbf{1}_n u_j$ is an equilibrium by construction.

DYNAMICS OF CONTINUOUS SETS

$$X(t+1) = F(X(t)) \qquad F : P(\mathcal{X})^n \to P(\mathcal{X})^n$$

DYNAMICS OF CONTINUOUS SETS

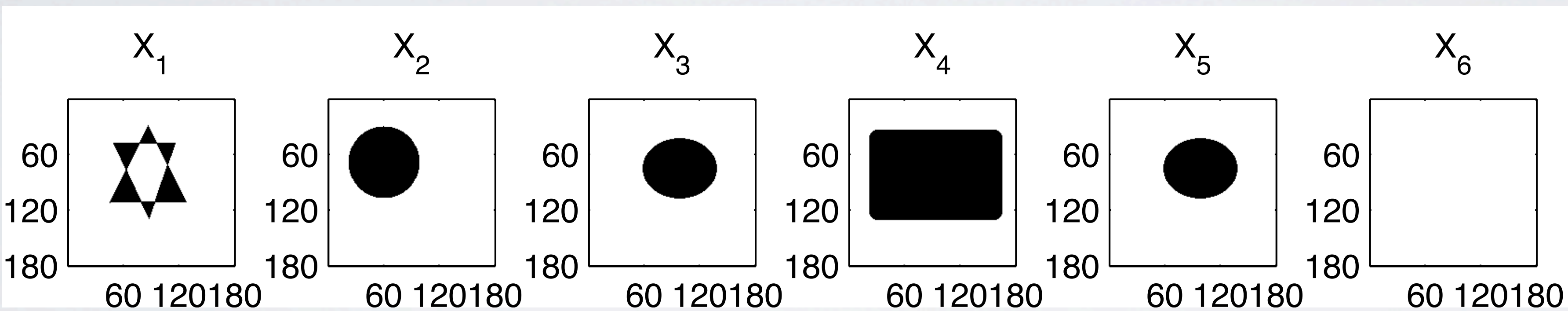
$$X(t+1) = F(X(t)) \qquad F : P(\mathcal{X})^n \to P(\mathcal{X})^n$$

- *Can we have complex behaviors, such as accumulation points, chaotic behaviors, when the domain* $\mathcal{X}$ *is infinite?*

DYNAMICS OF CONTINUOUS SETS

$$X(t+1) = F(X(t)) \qquad F : P(\mathcal{X})^n \rightarrow P(\mathcal{X})^n$$

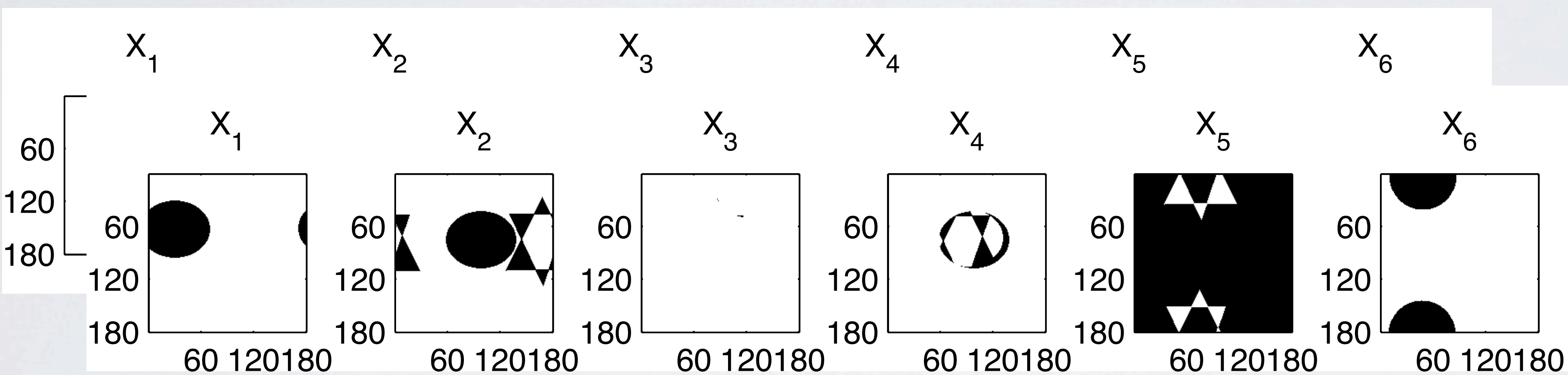
- *Can we have complex behaviors, such as accumulation points, chaotic behaviors, when the domain* $\mathcal{X}$ *is infinite?*



DYNAMICS OF CONTINUOUS SETS

$$X(t+1) = F(X(t)) \qquad F : P(\mathcal{X})^n \to P(\mathcal{X})^n$$

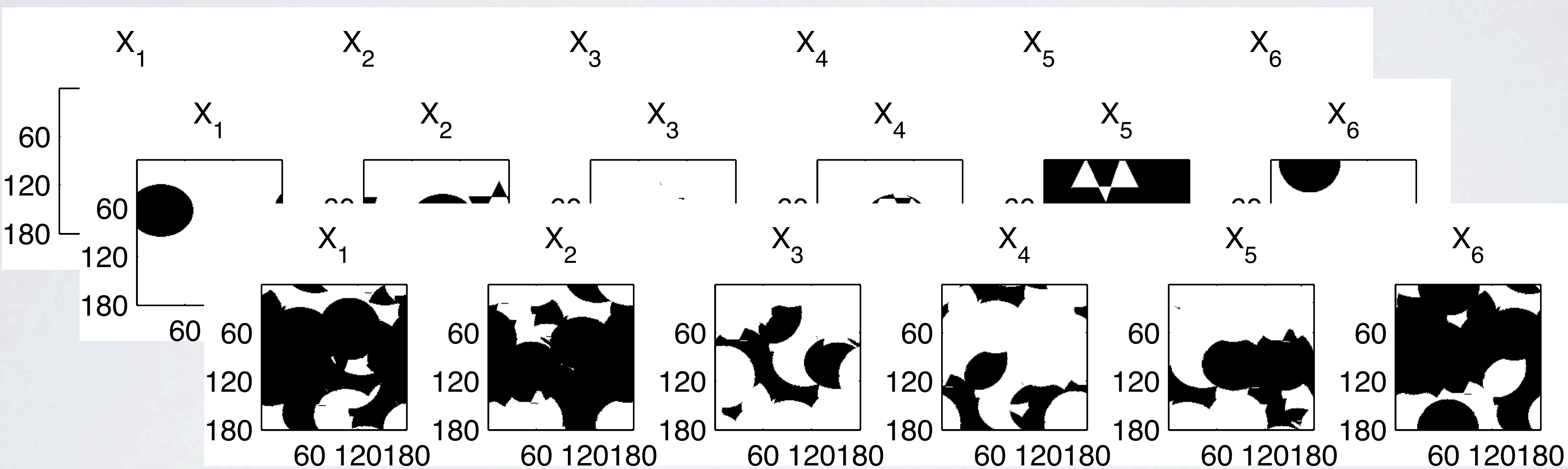
- *Can we have complex behaviors, such as accumulation points, chaotic behaviors, when the domain* $\mathcal{X}$ *is infinite?*



DYNAMICS OF CONTINUOUS SETS

$$X(t+1) = F(X(t)) \qquad F : P(\mathcal{X})^n \to P(\mathcal{X})^n$$

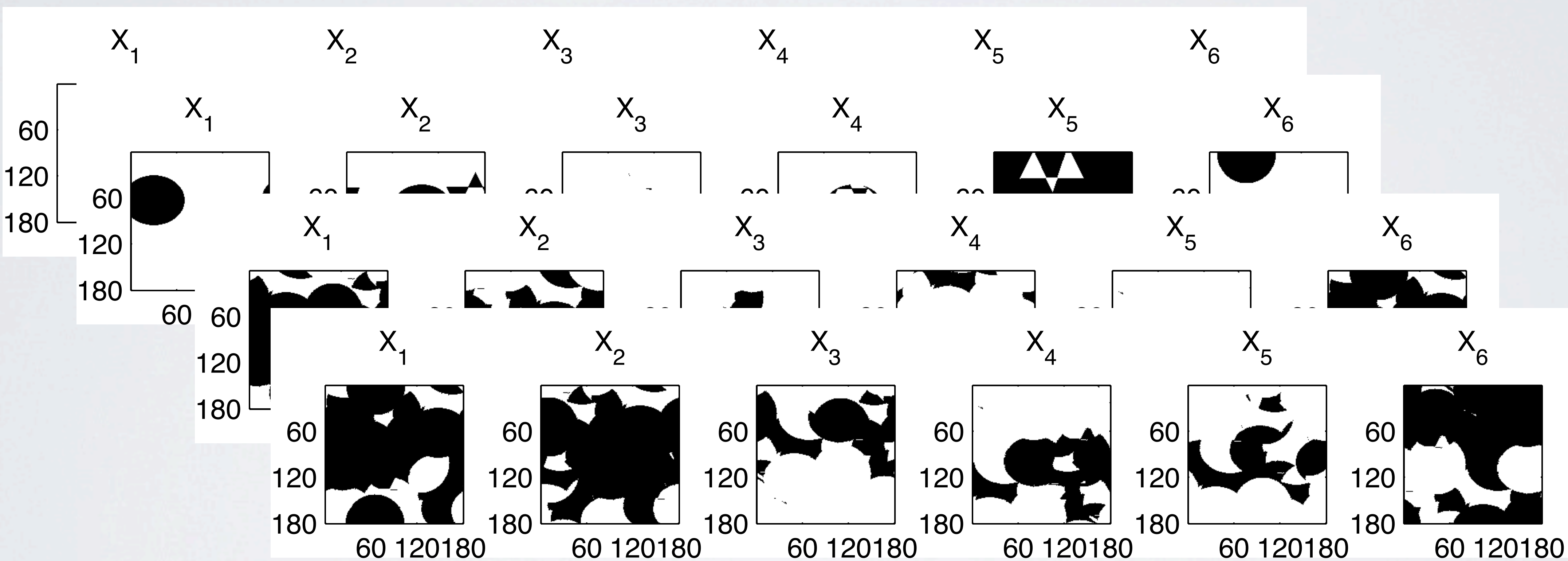
- *Can we have complex behaviors, such as accumulation points, chaotic behaviors, when the domain* $\mathcal{X}$ *is infinite?*



DYNAMICS OF CONTINUOUS SETS

$$X(t+1) = F(X(t)) \qquad F : P(\mathcal{X})^n \to P(\mathcal{X})^n$$

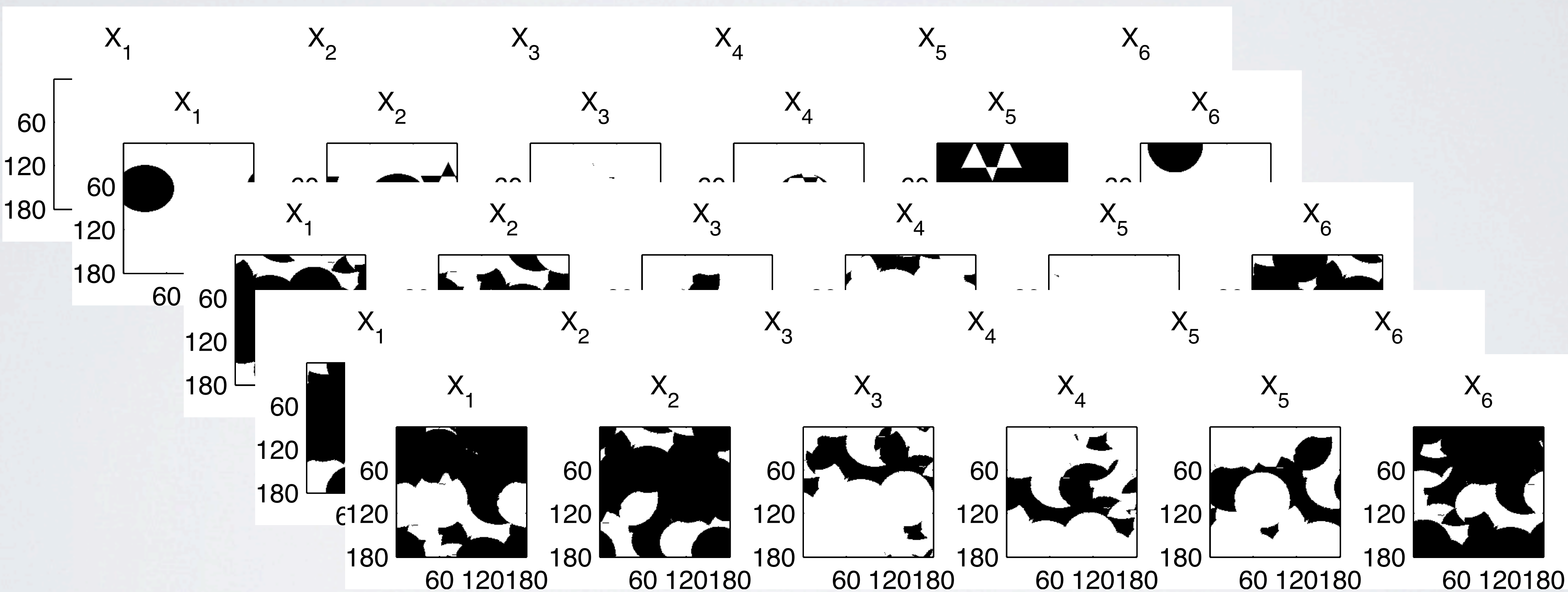
- *Can we have complex behaviors, such as accumulation points, chaotic behaviors, when the domain* $\mathcal{X}$ *is infinite?*



DYNAMICS OF CONTINUOUS SETS

$$X(t+1) = F(X(t)) \qquad F : P(\mathcal{X})^n \to P(\mathcal{X})^n$$

- *Can we have complex behaviors, such as accumulation points, chaotic behaviors, when the domain $\mathcal{X}$ is infinite?*



- *Establishing consensus on set-valued data may be much more difficult*

SET-VALUED BOOLEAN DYNAMICS

- A Boolean Algebra (BA) is $\Big(\underbrace{\tilde{\mathcal{B}}}_{\text{domain}}, \underbrace{\wedge}_{\text{meet}}, \underbrace{\vee}_{\text{join}}, \underbrace{\neg}_{\text{complement}}, \underbrace{0}_{\text{null}}, \underbrace{1}_{\text{unity}}\Big)$

$$a \vee (b \vee c) = (a \vee b) \vee c,\ a \wedge (b \wedge c) = (a \wedge b) \wedge c \text{ (associativity)}$$

$$a \vee b = b \vee a,\ a \wedge b = b \wedge a \text{ (commutativity)}$$

$$\vdots$$

SET-VALUED BOOLEAN DYNAMICS

- A Boolean Algebra (BA) is $(\underbrace{\tilde{\mathcal{B}}}_{\text{domain}}, \underbrace{\wedge}_{\text{meet}}, \underbrace{\vee}_{\text{join}}, \underbrace{\neg}_{\text{complement}}, \underbrace{0}_{\text{null}}, \underbrace{1}_{\text{unity}})$

$$a \vee (b \vee c) = (a \vee b) \vee c,\ a \wedge (b \wedge c) = (a \wedge b) \wedge c \text{ (associativity)}$$
$$a \vee b = b \vee a,\ a \wedge b = b \wedge a \text{ (commutativity)}$$
$$\vdots$$

- Every BA is isomorphic to an algebra of sets (Representation Theorem, Stones'36)

SET-VALUED BOOLEAN DYNAMICS

- A Boolean Algebra (BA) is $(\underbrace{\tilde{\mathcal{B}}}_{\text{domain}}, \underbrace{\wedge}_{\text{meet}}, \underbrace{\vee}_{\text{join}}, \underbrace{\neg}_{\text{complement}}, \underbrace{0}_{\text{null}}, \underbrace{1}_{\text{unity}})$

$$a \vee (b \vee c) = (a \vee b) \vee c,\ a \wedge (b \wedge c) = (a \wedge b) \wedge c \text{ (associativity)}$$

$$a \vee b = b \vee a,\ a \wedge b = b \wedge a \text{ (commutativity)}$$

$$\vdots$$

- Every BA is isomorphic to an algebra of sets (Representation Theorem, Stones'36)

Theorem: A Boolean dynamic system $X(t+1) = F(X(t))$, defined over an n–dimensional vector space $P(\mathcal{X})^n$ can be *simulated* by a suitable 2^n–dimensional logical iteration system

$$\tilde{x}(t+1) = \tilde{f}(\tilde{x}(t))\,.$$

SET-VALUED BOOLEAN DYNAMICS

- A Boolean Algebra (BA) is $(\underbrace{\tilde{\mathcal{B}}}_{\text{domain}}, \underbrace{\wedge}_{\text{meet}}, \underbrace{\vee}_{\text{join}}, \underbrace{\neg}_{\text{complement}}, \underbrace{0}_{\text{null}}, \underbrace{1}_{\text{unity}})$

$$a \vee (b \vee c) = (a \vee b) \vee c,\ a \wedge (b \wedge c) = (a \wedge b) \wedge c \text{ (associativity)}$$
$$a \vee b = b \vee a,\ a \wedge b = b \wedge a \text{ (commutativity)}$$
$$\vdots$$

- Every BA is isomorphic to an algebra of sets (Representation Theorem, Stones'36)

Theorem: A Boolean dynamic system $X(t+1) = F(X(t))$, defined over an n–dimensional vector space $P(\mathcal{X})^n$ can be *simulated* by a suitable 2^n–dimensional logical iteration system
$$\tilde{x}(t+1) = \tilde{f}(\tilde{x}(t))\,.$$

Implications:
- *F may possess only equilibria or cycles*
- *convergence study of* $\tilde{f}$ *allows us to conclude also on F*

GEOGRAPHICAL CHART ESTIMATION

- A large landscape environment

- Many robots with partial and noisy snapshots

$$X_i(0) = (I_i(0), V_i(0))$$

GEOGRAPHICAL CHART ESTIMATION

- A large landscape environment

- Many robots with partial and noisy snapshots

$$X_i(0) = (I_i(0), V_i(0))$$

- A global, consistent chart must be reconstructed only by local interaction

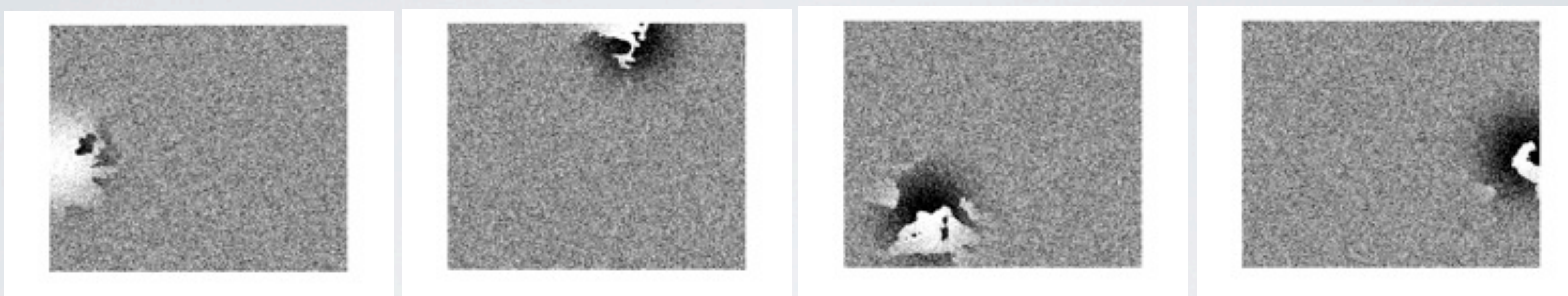
$$\begin{cases} I_i(t+1) & = & \bigcup_{H \in S_{\gamma+1}(C_i)} \bigcap_{h \in H} \left(V_h(t) \, \cap \, I_h(t)\right) \, , \\ V_i(t+1) & = & \bigcup_{H \in S_{\gamma+1}(C_i)} \bigcap_{h \in H} \, V_h(t) \, . \end{cases}$$

- Under suitable visibility and connectivity conditions, the image of the landscape is the unique equilibrium.

GEOGRAPHICAL CHART: SIMULATION

- Evolution of some robots' states

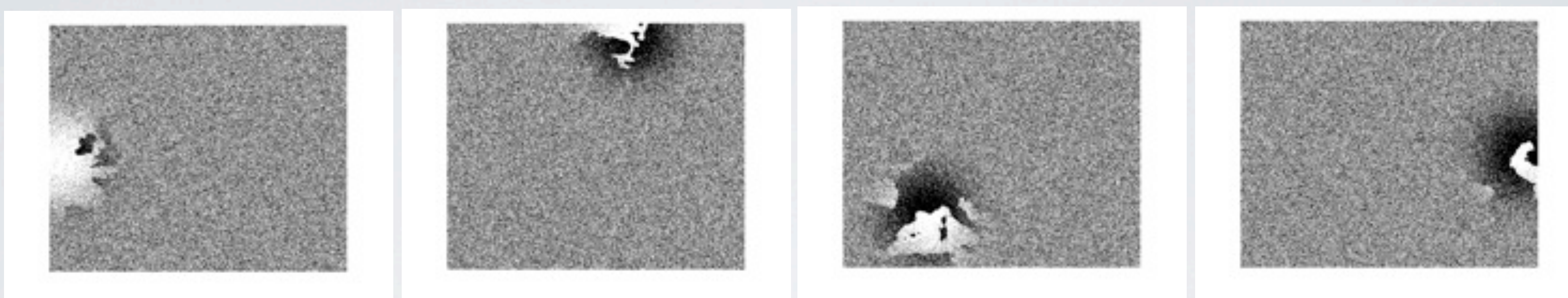
$t = 0$



GEOGRAPHICAL CHART: SIMULATION

- Evolution of some robots' states

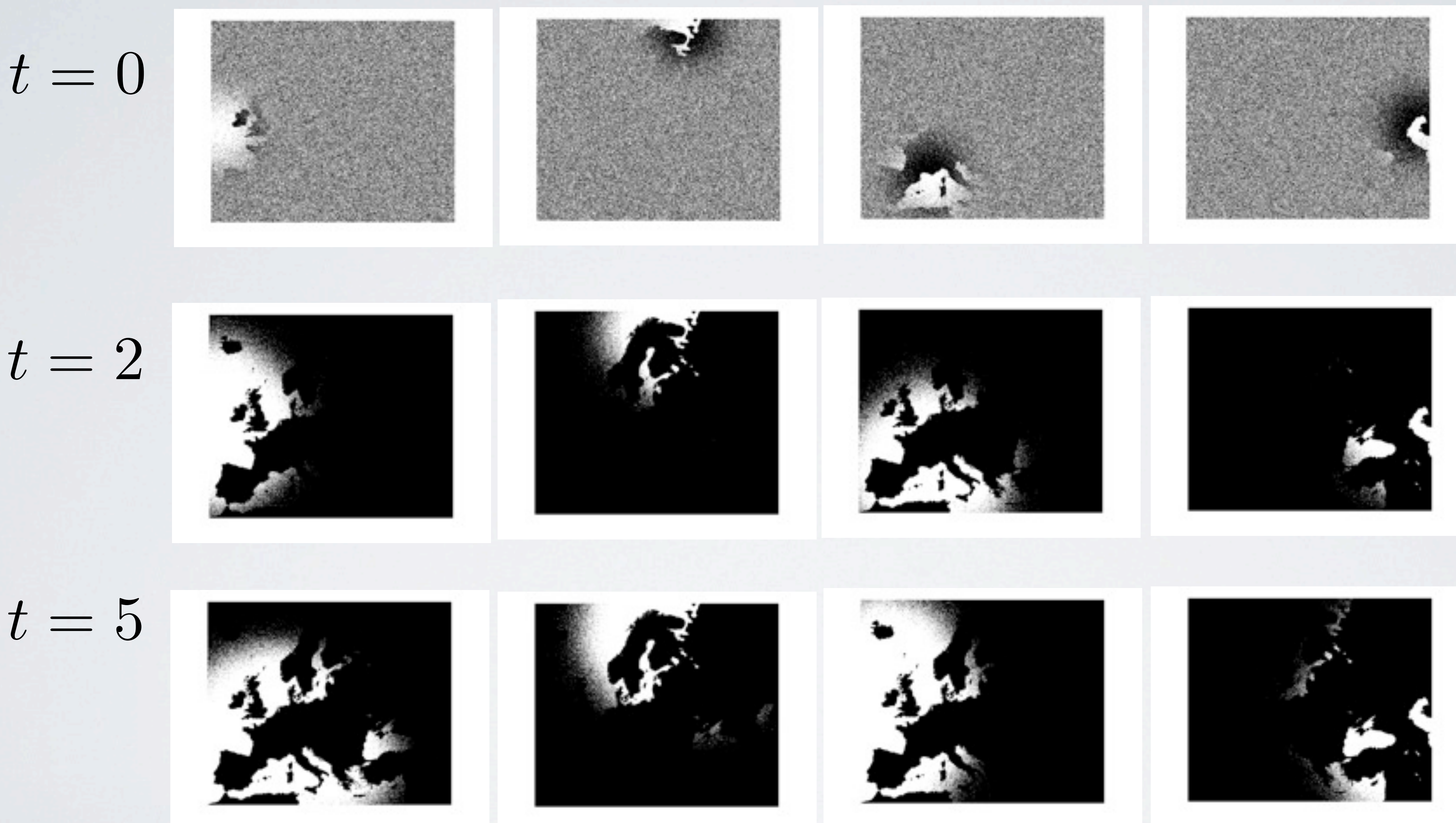
$t = 0$



$t = 2$

GEOGRAPHICAL CHART: SIMULATION

- Evolution of some robots' states



GEOGRAPHICAL CHART: SIMULATION

- Evolution of some robots' states

Workshop on New frontiers of Robotics -
Interdep. Research Center "E. Piaggio"
June 21-22, 2012 - Pisa (Italy)

BOOLEAN CONSENSUS FOR SOCIETIES OF ROBOTS

Adriano Fagiolini

DIEETCAM, College of Engineering, University of Palermo
Interdep. Research Center "E. Piaggio", University of Pisa