Local Motion Planning for Robotic Race Cars*

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***Abstract*— In this paper, we consider a motion planning problem for a robotized racing car. The car is approaching a sector of track and the goal of our algorithm is to identify the sequence of maneuvers that allows the car to complete the sector in minimum time. In the first part of the paper we study the physics of the problem and identify the optimal sequence of maneuvers taking into account also the boundaries of the track. In the second part, we consider a simplified sequence of maneuvers that is easier to implement on a low cost vehicle and that remains reasonably close to the optimal solution. For the simplified scenario geometrical considerations will be used to characterized the optimal maneuvers on both straight and bend sectors.**

I. INTRODUCTION

One of the problems that researchers and engineers have to face for autonomous driving of cars is the development of adequate algorithms for trajectory planning. The problem is particularly challenging when the car moves along a crowded and partially known urban or extra-urban roads [1], [2], a situation frequently considered in different research activities. In this paper, we address the same problem in a radically different scenario: a robotic racing car that runs on a track. The environment is well–structured and the only external presence allowed are other racing cars.

As a first step toward a comprehensive solution of motion planning for race cars, we consider here the situation when the car runs in isolation (or equivalently when the opponents are far away). In this case, the goal of the motion planner reduces to finding the trajectory that minimises the time to complete a given number of laps. However, we aim for a solution strategy that can be easily generalised to the situation where multiple cars are on the track and compete. In this case, several candidate trajectories have to be generated in a game theoretical setting (i.e., to consider possible overtake manoeuvres or to "cover" the best trajectories to the opponent). Therefore, numeric efficiency in the computation of the motion is just as important as the resulting performance on the lap. We advocate a solution approach that, in our belief, strikes a good compromise between performance and computation time. The idea is described in a different paper [3] and is based on the decomposition of the track into sectors and of each sector into cells. The trajectory is synthesised in two steps. First, a "local plan" is synthesised to connect the centres of two cells located at the extreme of a sector for any initial and final velocities in a discrete input set. As a result, we construct a graph where nodes are associated with cells and arcs between two nodes are associated with the local plan and have a cost given by the minimum time to complete it. Second, the global plan is synthesised by solving an optimisation algorithm on the graph.

In this paper we focus on the local planning problem, while the overall approach is described in a different paper [3]. The problem addressed can be summarised in the following terms: *find the set of manoeuvres that steer the car in minimum time between two configurations, each one identified by position on the track and velocity, respecting the dynamic constraints of the car and the geometric constraint of the track*. The literature of optimal (shortest) paths stems mainly from the seminal works on unicycle vehicles with a bounded turning radius by Dubins [4] and on the car moving both forward and backward by Reeds and Shepp [5]. Other optimization cost functions has been considered such as the minimum wheel rotation paths for differential-drive robots ([6]), minimum time trajectory ([7] for differential drive robots, [8] for omnidirectional vehicles, [9] for a mobile robot with a trailer subject to limited control inputs, and [10] for robots with two independently driven wheels), minimum path length ([11] and [12] for differential drive robot with limited Field-of-View and [13] for a car–like robot).

The model adopted in this paper is inspired to the one proposed for Stanford's Stanley autonomous car [14] which offers a sufficient coverage of the most important physical phenomena that are usually considered in this case. It comprises two different components: one is kinematic and the other one dynamic. The authors approach the trajectory tracking problem by decomposing it into two sub-problems: a steering controller, acting on the steering wheels angle, for path following; a cruise controller, acting on the engine power, to track desired velocity profiles. While the cruise controller has been synthesized in a standard way adopting PID control, the steering angle is firstly determined on the kinematic model, then it is modified to take into account dynamic effects described by the dynamic model. We follow here the same rationale, although the steering controller is slightly modified to derive a kinematic model that can be effectively used to synthesise the optimal trajectories. Hence, the motion planning problem can be approached restricting to the kinematics of the vehicle as far as a controller is applied

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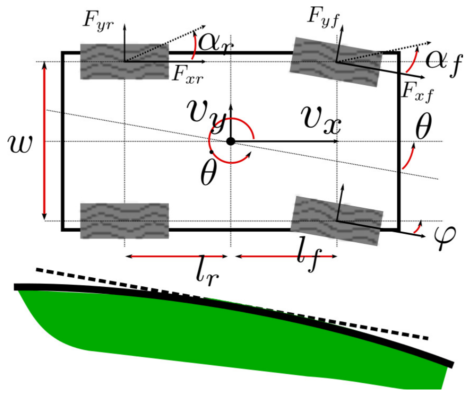

Fig. 1. Vehicle dynamic model

to the dynamics. Under the constraint that the car does not slip away (i.e. the lateral acceleration remains bounded below a given bound), we show that the optimal plan is composed of a concatenation of elementary manoeuvres.

We offer a precise characterisation of the alphabet of elementary manoeuvres and we provide a preliminary characterisation of the optimal sequences of manoeuvres on the track sectors. Due to the complexity of the problem, we consider a simplified sequence of maneuvers that is easier to implement on a low cost vehicle and that remains reasonably close to the optimal solution. The simplified manoeuvers allow us to use geometrical considerations to characterized the sub–optimal sequences of maneuvers on both straight and bend sectors.

II. Problem Definition

In this section we first introduce a race car model and a geometric characterisation of the track. We finally formulate the motion planning problem that will be analysed later on.

A. Car Model

Dynamic Model. Let $\langle W \rangle = \{O_w, X_w, Y_w, Z_w\}$ be a right–handed fixed (world) reference frame. The dynamic model is the *bicycle model* of an automobile [15], shown in figure 1. The model considers the effects of tires slip and steering servo motor, i.e.,

$$\begin{aligned}
\dot{v}_x &= \frac{F_{xr} + F_{xf}\cos(\varphi) - F_{yf}\sin(\varphi)}{m} + \dot{\theta}\, v_y, \\
\dot{v}_y &= \frac{F_{yr} + F_{xf}\sin(\varphi) + F_{yf}\cos(\varphi)}{m} - \dot{\theta}\, v_x, \\
\ddot{\theta} &= \frac{l_f\,(F_{xf}\sin(\varphi) + F_{yf}\cos(\varphi)) - l_r\, F_{yr}}{I_z}, \\
|\varphi| &\le \overline{\varphi},
\end{aligned}$$

where v_x and v_y are the longitudinal and lateral velocity at the Center of Mass (CoM) of the vehicle, φ is the steering angle, $\overline{\varphi}$ is the maximum steering angle and θ is the vehicle heading w.r.t. X_w axis. $L = l_f + l_r$ is the total wheel base, where l_f, l_r are the longitudinal distance from the CoM to the front and rear tires respectively (see Fig. 1). Each front and rear tire provides a lateral force F_{yf}, F_{yr} which is perpendicular to the rolling direction of the wheel and a longitudinal force F_{xf}, F_{xr} which is parallel to the rolling direction (the *single track model* is here adopted [16]). The lateral forces are

$$\begin{aligned}
F_{yf} &\approx -c_y\,\alpha_f, \\
F_{yr} &\approx -c_y\,\alpha_r,
\end{aligned}$$

where the approximation is due to the linearisation for dry asphalt of the nonlinear relations and c_y is the tire cornering stiffness. α_f and α_r are the tire sleep angles on the front and the rear wheels, respectively, and are given by

$$\begin{aligned}
\alpha_f &= \arctan\left(\frac{v_y + \dot{\theta}\, l_f}{v_x}\right) + \varphi, \\
\alpha_r &= \arctan\left(\frac{v_y - \dot{\theta}\, l_r}{v_x}\right).
\end{aligned}$$

Similarly, the longitudinal forces are

$$\begin{aligned}
F_{xf} &\approx c_x\,\sigma_f, \\
F_{xr} &\approx c_x\,\sigma_r,
\end{aligned} \tag{1}$$

where the approximation is again due to linearization and c_x is the tires longitudinal stiffness. σ_f and σ_r are the tire slip ratios on the front and the rear wheels, respectively, and are given by

$$\sigma_i = \begin{cases} \dfrac{r_{eff}\,\omega_{w,i} - v_x}{v_x} & \text{when breaking,} \\ \dfrac{r_{eff}\,\omega_{w,i} - v_x}{r_{eff}\omega_{w,i}} & \text{when accelerating,} \end{cases}$$

where $i \in \{r, f\}$, r_{eff} is the effective radius of the wheel and $\omega_{w,i}$ is the wheel angular velocity.

Notice that c_y and c_x can have different values if each wheel is driven independently. Moreover, for a 4WD vehicle (as the car considered in this paper), the same ω_{w_i} is applied to each tire, hence we will use ω_w henceforth. Additionally, $\sigma_f = \sigma_r = \sigma$ for a 4WD.

Kinematic Model for Motion Planning. The kinematic model described in [14] uses a *forward driving* model (front traction). The main difference of our kinematic model is that we opted for a rear traction model for the vehicle.

Based on the dynamic model found in [16], we further modified the Stanley's dynamic model by adding an aerodynamic force effect to the vehicle (F_{aero}) given by

$$F_{aero} = c'_a\, v_x^i$$

where c'_a is the aerodynamic drag coefficient and $i = 1$ for air laminar motion, while $i = 2$ for turbulent motion. For the sake of simplicity, this paper considers the laminar motion is considered. The laminar regime is accurate enough for scale car models used in robotics laboratories. However, since the computation of the extremals is not affected by the regime of the fluid motion (see section III), we believe that the adaptation of the approach to turbulent regime can be made with little effort. Thus, the complete model is given by:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{v}_x\\ \dot{v}_y\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} v_x\cos(\theta) - v_y\sin(\theta)\\ v_x\sin(\theta) + v_y\cos(\theta)\\ \frac{F_{xr}+F_{xf}\cos(\varphi)-F_{yf}\sin(\varphi)}{m} + \dot{\theta}\, v_y - \frac{F_{aero}}{m}\\ \frac{F_{yr}+F_{xf}\sin(\varphi)+F_{yf}\cos(\varphi)}{m} - \dot{\theta}\, v_x\\ \frac{l_f\,(F_{xf}\sin(\varphi)+F_{yf}\cos(\varphi))-l_r\,F_{yr}}{I_z}, \end{bmatrix}$$

where the inputs are the steering angle φ and the wheel angular velocity ω_w resulting from the application of the forces F_{xf} and F_{xr}, see (1). Notice that x, y and θ are expressed in the global reference frame, while v_x and v_y are expressed in the car moving frame attached to the CoM (see Figure 1). In particular, x and y are the coordinates of the midpoint of the rear axle (see Figure 2), m is the mass of the vehicle and I_z is the moment of inertia about the vertical axis. The dynamic model thus presented can be further broadened adding, for example, the wind effect or the rolling resistance of the tires. However, such extensions are postponed to future works.

Assuming that the steering and thrust controller proposed by Hoffman et al. [14] are used, the extended kinematic model adopted for optimal path synthesis is given by

$$\dot{q} = \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{v}_x\end{bmatrix} = \begin{bmatrix} v_x\cos(\theta)\\ v_x\sin(\theta)\\ v_x\,\frac{\tan(\varphi)}{L}\\ a - c_a v_x\end{bmatrix} \tag{2}$$

where φ is the steering angle, one input of the model, constrained by $|\varphi| \leq \overline{\varphi}$, $c_a = \frac{c_a'}{m}$ and a is the acceleration input, given by

$$a = \frac{F_{xr} + F_{xf}\,\cos(\varphi) - F_{yf}\sin(\varphi)}{m},$$

that, due to the limited steering angle, simplifies to

$$a \approx \frac{F_{xr} + F_{xf}}{m},$$

while the term $\dot{\theta}v_y$ is neglected since it is supposed to be compensated by the traction controller.

B. Track Description

In our framework, the track is composed of straight sectors and of sectors comprising a bend on the right (or, equivalently, on the left), as shown in Figure 2.

The track is characterised by two boundaries, left B_l and right B_r, which are also formed by a sequence of a straight line, arc of circle and another straight line. We make the simplifying assumption that the distance between B_l and B_r is constant along the track with width $W \geq 0$.

For the sectors containing a bend the two boundaries are parametrized as show in Fig. 2. The angle γ is the characteristic curve angle, i.e. the angle between the two straight line sectors of each boundary. The bend boundaries are characterized by the centers C_b and the radii R_b. In particular, R_b^i refers to the inner boundary, while R_b^o to the outer. Trivially, $R_b^i < R_b^o$ for widths $W > 0$.

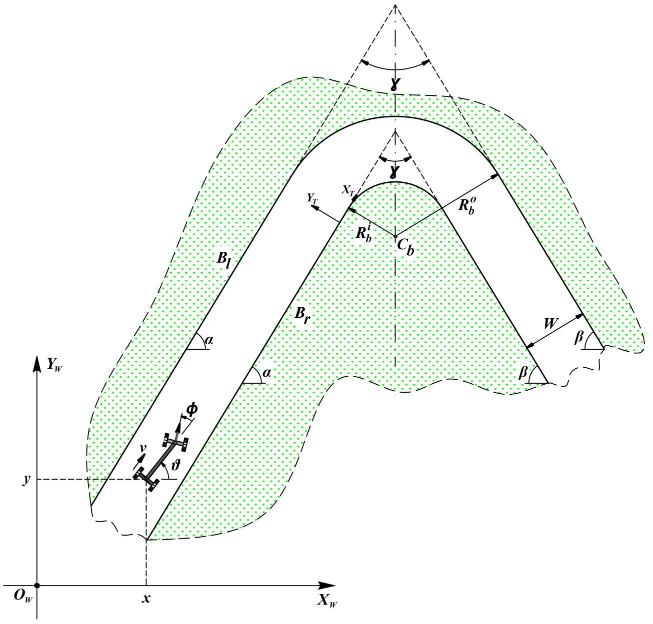

Fig. 2. Mobile robot and system coordinates, with an example of a track.

Without loss of generality we can assume the bend to be oriented such that the axis Y_w is parallel to the the bisector of γ. The angles α and $\pi - \beta$ are the orientations of the line sectors w.r.t. X_w.

Reference frames. Let $\langle T\rangle = \{O_T, X_T, Y_T, Z_T\}$ be the *track reference frame* with O_T a point on B_r, X_T tangent to B_r and Y_T pointing towards the left boundary B_l (see Fig. 2). This frame is a Frenet frame attached on the the right boundary of the track. Consider the *track initial reference frame* $\langle I\rangle$ as $\langle T\rangle$ with O_T placed on B_r at the beginning of the track and the *track final reference frame* $\langle E\rangle$ as $\langle T\rangle$ with O_T placed on B_r at the end of the track.

A point ${}^{T}p_r$ on X_T has a corresponding point on B_l given by ${}^{T}p_l = {}^{T}p_r + [0,\, W]^T$. In particular, any point inside the track at the same distance of ${}^{T}p_r$ from the curve can be expressed as ${}^{T}p = {}^{T}p_r + k[0,\, W]^T$, where $0 \leq k \leq 1$. We will denote by $\mathcal{P}$ the set of all points lying inside the track, for which there exist a Frenet frame in which their X coordinate is 0 and their Y coordinate is smaller than W: $\mathcal{P} = \big\{p|\exists\langle T\rangle \text{ s.t. } {}^{T}p = k[0,\, W]^T, 0 \leq k \leq 1\big\}$.

C. The optimal path planning problem

The goal of the paper is to find an optimal path that steers the car from a configuration q_I (associated with a cell at the beginning of a sector) to a configuration q_E (associated with a cell at the end of the sector) in minimum time. Let t_I represent the instant when the planned motion starts and t_E the instant when the motion ends. A configuration q is associated to the state variables $q = [x,\, y,\, \theta,\, v_x]$. The solution of the motion planning problem aims to identify the acceleration function $a(t)$, the steering angle function $\varphi(t)$ and the final instant t_E. In mathematical terms the problem can be formulated as the following optimal control problem:

Problem 1: Optimal Control Problem for Local Planning

$$\min_{a(t),\varphi(t)} \int_{t_I}^{t_E} L(q,a,\varphi)dt, \text{ subject to}$$

(1) $q(t)$ solution of (2),
(2) $q(t_I) = q_I, q(t_E) = q_E$, $\forall t \in [t_I, t_E]$
(3) $q(t) \in \mathcal{Q}$
(4) $v_x^2(t)|\tan\varphi| \le a_l L$
(5) $a(t) \in [\underline{a}, \overline{a}], \varphi(t) \in [-\overline{\varphi}, \overline{\varphi}]$.

The couple of constraints (2) is on the initial and on the final desired configurations. The constraint (3) requires that all configurations throughout the interval $[t_i, t_E]$ remain feasible. To elaborate this notion, for a given configuration q define by $p_r(q)$ the subvector $p_r = [x, y]$ associated with the mide point of the rear axle. The position of the midpoint of the front axle is given by $p_f(q) = [x + L\cos\theta, y + L\sin\theta]$. The set of feasible configurations $\mathcal{Q}$ is made of all configurations such that both p_r and p_f are inside the track: $\mathcal{Q} = \{q|p_f(q), p_r(q) \in \mathcal{P}\}$. Clearly both the initial and the final configuration q_I and q_E are required to be inside $\mathcal{Q}$ and so have to be the intermediate configurations. The constraint (4) requires that the car never exceeds the maximum allowed lateral acceleration a_l. This gives rise to the following constraints for the intermediate configurations of the system:

$$v_x \le \sqrt{a_l R} \tag{3}$$

where $R = \frac{L}{|\tan\varphi|}$. Notice that a_l is a function of the tires grip, which depends on the ground characteristics, e.g., dry or wet asphalt, off road, etc., and generates constraint depending on the state variable v_x and the control input φ. The constraint (5) is on the physical limitation of the vehicle (maximum and minimum acceleration and steering angle). The cost function is in this case the time to complete the motion:

$$\int_{t_I}^{t_E} L(q,a,\varphi)dt, \tag{4}$$

where $L(q, a, \varphi) = a_1$, with $a_1 > 0$.

III. Maneuver Extremals

In order to find a solution to the problem 1, we first need to identify the set of extremals that verify necessary condition for optimality based on the Pontryagin Minimum Principle (see e.g. [17] and [18] for the constraints on the state and control variables). For this purpose we study the problem disregarding the constraints on the configurations q that impose that the vehicle is on the track (constraint (3)). We will recover such geometric constraints later on.

Proposition 1: Optimal paths consists of concatenation of

1) Straight line $\mathcal{S}$, travelled with any velocity profile, compatible with the maximum and minimum accelerations, i.e. $\overline{a}$ and $\underline{a}$, respectively;
2) Circular curve $\mathcal{C}_{\overline{r}}$ travelled with constant maximum velocity $\overline{v_x} = \frac{\overline{a}}{c_a}$; the radius is fixed to the maximum value $\overline{r}$ that is compatible with the constraint on the lateral acceleration: $\overline{r} = \overline{v_x}^2/a_l$, where a_l is the maximum lateral acceleration;
3) Circular curve $\mathcal{C}_{\underline{r}}$ travelled at (possibly time–varying) velocity $0 \le v_x \le \sqrt{\frac{a_l L}{|\tan(\overline{\varphi})|}} = v_{x\varphi}$; the radius is fixed to the minimum possible value allowed by the vehicle : $\underline{r} = \frac{L}{|\tan(\overline{\varphi})|}$;
4) Variable radius curves $\mathcal{V}_{\overline{a}}$ and $\mathcal{V}_{\underline{a}}$ executed with maximum or with minimum acceleration respectively, which always verify the relation $v_x^2|\tan(\varphi)| = a_l L$.

Proof: The state-control constraint $v_x^2|\tan\varphi| \le a_\ell L$ generates two constraints

$$C_1(q,a,\varphi) = v_x^2 \tan\varphi - a_\ell L \le 0, \quad \varphi > 0$$
$$C_2(q,a,\varphi) = -v_x^2 \tan\varphi - a_\ell L \le 0, \quad \varphi < 0\,.$$

Hence, the Hamiltonian function associated to the optimal control problem 1 is

$$H = a_1 + \lambda_1 v_x \cos\theta + \lambda_2 v_x \sin\theta + \lambda_3 \tan\varphi \frac{v_x}{L} + \lambda_4(a - c_a v_x) + \mu_1(v_x^2\tan\varphi - a_l L) - \mu_2(v_x^2 \tan\varphi + a_l L) \tag{5}$$

where $\mu_1 = 0$ ($\mu_2 = 0$) when $v_x^2\tan\varphi < a_l L$ ($-v_x^2\tan\varphi < a_l L$).

From the dynamic of the co-state $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ we have $\dot\lambda_1 = -\frac{\partial H}{\partial x} = 0$ and $\dot\lambda_2 = -\frac{\partial H}{\partial y} = 0$ hence we can define $\lambda_1 = d\cos\gamma$ and $\lambda_2 = d\sin\gamma$ obtaining

$$\begin{aligned} H = a_1 + d\,v_x\cos(\theta-\gamma) + \lambda_3\tan\varphi\frac{v_x}{L} + \\ + \mu_1(v_x^2\tan\varphi - a_l L) - \mu_2(v_x^2\tan\varphi + a_l L) \end{aligned} \tag{6}$$

the dynamics of the rest of the co-state are

$$\begin{aligned} \dot\lambda_3 &= -\frac{\partial H}{\partial\theta} = d\,v_x\sin(\theta-\gamma) \\ \dot\lambda_4 &= -\frac{\partial H}{\partial v_x} = -d\,\cos(\theta-\gamma) - \lambda_3\frac{\tan\varphi}{L} + \\ &\qquad - c_a\lambda_4 - 2(\mu_1-\mu_2)v_x\tan\varphi \end{aligned} \tag{7}$$

We first analyze the extremal arcs when the state-control constraint is active, i.e. we assume $v_x^2|\tan\varphi| = a_l L$. Without loss of generality, we consider only the case in which $C_1 = 0$ and hence $C_2 < 0$. The opposite case gives a similar result. Notice that constraints C_1 and C_2 are mutually exclusive. With this choice, $\mu_1 \ge 0$ and $\mu_2 = 0$. Moreover, if $C_1 = 0$ the control variable φ can be obtained from the state variable v. However, φ could be less than $\bar\varphi$. The remaining control variable is $a \in [\underline{a}, \overline{a}]$.

If $-\bar\varphi < \varphi < \bar\varphi$ and $\underline{a} < a < \overline{a}$ than, for optimality, we have $\frac{\partial H}{\partial\varphi} = 0$, i.e. $v(\lambda_3/L + \mu_1 v_x)(1 + \tan\varphi^2) = 0$, and hence, as $v_x > 0$, $\mu_1 = -\lambda_3/(L v_x)$ which implies $\lambda_3 \le 0$. Moreover, we have $\frac{\partial H}{\partial a} = 0$, i.e. $\lambda_4 = 0$, and hence $\dot\lambda_4 = 0$. From the second equation in (7), $\lambda_3 = -\frac{dv_x^2}{a_l}\cos(\theta-\gamma)$. Deriving λ_3 and considering the first equation in (7) we obtain that $a = c_a v_x + a_l\tan(\theta-\gamma)$. However, for minimum time problem, a necessary condition for optimality implies that $\dot H = 0$ along the optimal trajectories. By substituting all the above results, the Hamiltonian becomes $H = 2dv_x\cos(\theta-\gamma)$ and its derivative is $\dot H = da_l\sin(\theta-\gamma) = 0$ which implies $\theta = \gamma = const.$. This conclusion violates the assumption, i.e. $C_1 = 0$. As a consequence, at least one of controls a and φ must to be on the boundary.

If $0 < v_x < \overline{v_x}$ and $a = \overline{a}$ ($a = \underline{a}$) v_x increases up to $\overline{v_x}$ (decreases down to 0) and φ changes with v_x according to $C_1 = 0$, i.e. $v_x^2 \tan\varphi = a_l L$. Notice that the constraint on the maximum lateral acceleration can be written in terms of the curvature radius R as $v_x^2 = Ra_l$, hence the extremal is a curve with a radius that increases proportionally to v_x^2, i.e. an arc of type $\mathcal{V}_{\overline{a}}$ (or $\mathcal{V}_{\underline{a}}$).

If $v_x = \overline{v_x}$ we have $\dot{v}_x = 0$ and hence $a = c_a\overline{v_x} = \overline{a}$. In this case the extremal arc is an arc of circle with radius $\frac{L}{\tan\varphi}$ followed at constant velocity $\overline{v_x}$ with an angle φ that is solution of $v_x^2 \tan\varphi = a_l L$, i.e. an arc of type $\mathcal{C}_{\bar{r}}$. This extremal exists only if $\overline{\varphi} \geq \arctan\frac{a_l L}{\overline{v_x}^2}$. In other words if the curves of the track are not too sharp this extremal may be an arc of the optimal path. In case of sharp turns it implies that the velocity must be decreased before the turn to avoid a lateral acceleration larger than a_l.

If $-\overline{\varphi} \leq \varphi \leq \overline{\varphi}$ we have $\frac{\partial H}{\partial \varphi} = \lambda_3(1+\tan^2\varphi)\frac{v_x}{L} = 0$. This implies $\lambda_3 = \dot{\lambda}_3 = 0$ and, from the first equation in (7) we have $\theta = \gamma$, $\dot{\theta} = 0$ and hence $\varphi = 0$. The obtained extremal is hence a straight line that is part of an optimal path only if is followed at the maximum speed based on the initial and final values of the velocity, i.e. an arc of type S.

If $\varphi = \pm\overline{\varphi}$ the vehicle proceeds along a arc of circle of constant radius $\underline{r} = \frac{L}{|\tan\varphi|}$. The arc is part of an optimal path only if is followed at the velocity $0 \leq v_x \leq \sqrt{\frac{a_l L}{|\tan\varphi|}}$, i.e. an arc of type $C_{\underline{r}}$.

■

A. Extremals with Geometric Constraints

For the principle of optimality any subpath of an optimal path is optimal itself. When taking into account the physical borders of the track the optimal solution will consist of subpaths along such borders (named as constrained subpaths) and subpaths strictly verify the physical border constraints (named as unconstrained subpaths).

In a straight sector, the track border constraints generates straight constrained subpaths that are equivalent to the unconstrained ones. On the other hand, along a curve the only two constrained paths are the arcs of circle with radius R_o and R_i. Along the optimal path those constrained paths are concatenated with the extremal unconstrained paths. To be optimal the constrained subpath must be followed at the maximum allowed acceleration without violating the lateral acceleration constraint.

IV. Optimal Maneuvers

After introducing the alphabet of optimal manoeuvres (extremals) that compose the motion plan, we will now discuss the optimal sequence of manoeuvres to be used for the solution of Problem 1. We will discuss two different cases. In the first, the sector of the track between the two end-points of the path is a straight line (straight sector), while in the second it contains a bend (turn sector).

Generally speaking, the solution to this problem is given by a concatenation (a *word*) of extremals. Each extremal is associated with some free parameters. For instance parameters of the straight line are initial and final velocity and length. When two extremals are interconnected some of the free parameters are constrained (for instance the initial velocity of the second extremal has to be equal to the final velocity of the first one). Other constraints are obviously given by the two end-points that have to be interconnected by the sequence. Nevertheless, some of the parameters in the word remain free choice. So, in general, to find the optimal sequence of manoeuvres that steers the vehicle between two configurations, one has to identify the optimal sequence of extremals and the correct choice of parameters that produce a minimum time transition between the two configurations.

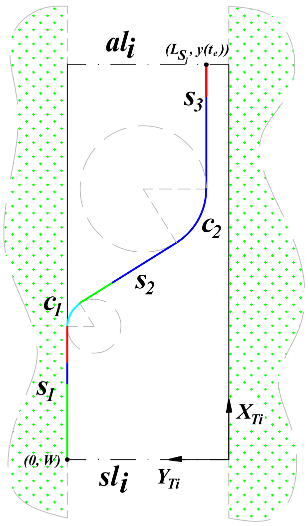

Fig. 3. Optimal solutions on a straight sector.

The solution of this problem is very challenging. However, by approximating the extremals with circular arcs, geometrical properties can be used to reduce the number of parameters to be identified. We will analyse in the depth this simplified case and show how it can be a useful source of inspiration for a solution heuristic that applies to the general case.

A. Sub–optimal sequences in a simplified case

In this section, we consider a simplified scenario in which the circular arcs, followed at constant speed with $v_x^2|\tan(\varphi)| = a_l L$, substitute $\mathcal{V}_{\overline{a}}$ and $\mathcal{V}_{\underline{a}}$ arcs. This leads to a sub–optimal path whose cost remains reasonably close to the optimal solution. We will analyse in depth this simplified case and show how it can be a useful source of inspiration for a heuristic solution that applies to the general case.

Each sector is delimited by two lines sl and al which are perpendicular to the lane. So the initial configuration $q(t_I)$ is such that the sub-vector $p_f(q(t_I)) \in sl$ (i.e., the front axle is on the start line) and the final configuration $q(t_E)$ is such that $p_f(q(t_E)) \in al$ (i.e. the front axle is on the finish line of the sector). For the sake of simplicity, we further assume that $\varphi(t_I) = \varphi(t_E) = 0$, i.e. the car is parallel to the lane at the beginning and the end of the sequence.

1) Trajectories for straight sectors: Consider a straight sector as represented in Figure 3. We are interested in

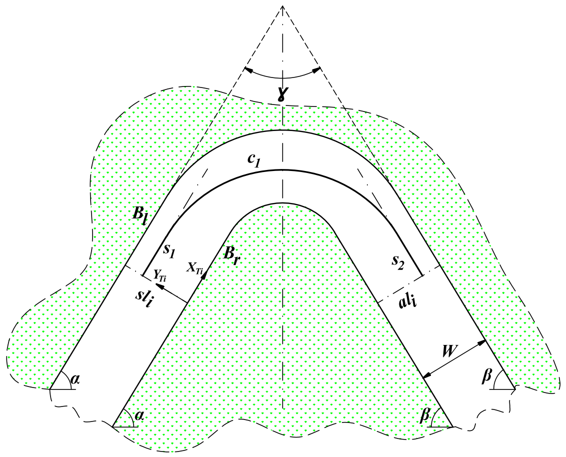

Fig. 4. Sub–optimal path on a turn sector.

analysing all position pairs $p(t_I) = [x(t_I), y(t_I)]$ and $p(t_E) = [x(t_E), y(t_E)]$ where $x(t_I) = 0$ and $x(t_E) = L_S$ (L_S is the length of the sector) and $0 \leq y(t_I), y(t_E) \leq W$. Given the optimal path from $p(t_I)$ to $p(t_E)$ with velocities $v(t_I) = v_I$ and $v(t_E) = v_E$, for translation invariance, it is also the optimal path from $(0, y(t_I)+h)$ to $(L_{S_i}, y(t_E)+h)$ with $0 \leq h \leq W - \max\{y(t_I), y(t_E)\}$ and with the same velocities. Furthermore, the symmetry with respect to lines parallel to the sector borders provide the optimal path from $[0, y(t_I)]$ to $[L_S, 2y(t_I) - y(t_E)]$ with $0 \leq y(t_I) \leq \frac{W+y(t_E)}{2}$ and with $v(t_I)$ and $v(t_E)$. Hence, without loss of generality, it is sufficient to consider $y(t_I) = W$ and $0 \leq y(t_E) \leq W$.

The analysis can be carried out using the classical arguments of Dubins [4]. In particular, excluding the case of L_S much smaller than the minimum radius of curvature $R = \frac{L}{\tan \bar{\varphi}}$ (which gives rise to more complex sequence), the optimal word is given by $\mathcal{SCSCS}$, where the turns $\mathcal{C}$ can be a circle with generic radius travelled at maximum allowed velocity and some of the extremals of the sequence can be missing (or equivalently have zero duration).

In the sequence $\mathcal{SCSCS}$ the free parameters are the length s_1 and s_3 of the first and of the third straight sectors and the velocities v_1 and v_2 of the two bends. By varying all the four parameters a path from $p(t_I)$ to $p(t_E)$ is found. Additional constraints come from geometric considerations: for example, it can be shown that the quantity $s_1 + s_3$ can never exceed L_S.

The parameters are strongly affected by the initial and the final velocities. For example, if $v(t_I) = v(t_E) = \bar{v}$ (and if L_S is sufficiently large with respect to the radius of the admissible curve at maximum speed $R = \frac{\bar{v}_x^2}{a_l}$) the solution is a Dubins path $\mathcal{CSC}$ with $s_1 = s_3 = 0$ and $v_1 = v_2 = \bar{v}_x$. It can be shown that, for small values of c_a and for sufficiently large L_S, the behaviour along the solution is to use maximum acceleration along the first two straight lines while braking (with minimum negative acceleration), if needed, along the third straight line to reach the desired final speed.

2) Trajectories for turn sectors: Consider a turn sector S_i as represented in Figure 4. We are interested in analysing all position pairs $p(t_I) = [x(t_I), y(t_I)]$ and $p(t_E) = [x(t_E), y(t_E)]$ where $x(t_I) = 0 \in sl_i$ and $x(t_E) \in al_i$ and $0 \leq y(t_I), y(t_E) \leq W$. Contrary to the straight sectors, in this case invariance properties of the optimal solution do not help restrict the initial of final points to be considered.

Based on the geometric characteristics of a turn sector and using arguments *à la* Dubins, the considered word is in this case given by $\mathcal{SCS}$. By using this word to move the car from one configuration to another, the only free parameter is s_1, which is the length of the first (short) straight line. The radius and hence the velocity v_1 associated to the curve is determined univocally by s_1. Indeed, there exists only a circle tangent to both the straight lines of $\mathcal{SCS}$ at distance s_1 from the initial point. Obviously, for the existence of an admissible velocity v_1 and for the sector borders constraints, the values of s_1 are limited. Finally, the length of the second (short) straight line follows from the position of the arrival line al,form s_1 and v_1. By varying the parameter l_1 the path from $p(t_I)$ to $p(t_E)$ can be found.

As for the case of the straight line, the solution is strongly affected by the initial and final velocities, as well as by the values of R_b^i and R_b^o. It can be shown that, for sufficiently large values of R_b^i, the behaviour along the solution is to touch the inner border of the turn.

B. Optimal sequences in the general case

In the general case, the family of the extremals is much richer and any two extremals can potentially be interleaved by a straight line. Therefore, we propose here a heuristic solution (rather than the exact one) that is mathematically tractable and is closely inspired to the simplified case discussed above.

First of all, we can regroup the maneuvers that move along the constraint imposed by the lateral acceleration, and the other that will not. Hence, we will define the maneuvers

$$\mathcal{T}_{\underline{a}} = \mathcal{C}_{\overline{r}} \circ \mathcal{V}_{\underline{a}}$$
$$\mathcal{T}_{\overline{a}} = \mathcal{V}_{\overline{a}} \circ \mathcal{C}_{\overline{r}}$$

Let us first focus on the straight sector and assume that the velocity at the beginning and at the end is higher than the one that can be held in the bend. In the simplified case, the manoeuvre (which is essentially a change of lane) clearly requires two circular curve in the opposite sense interleaved by a straight line. Besides, we have an initial straight and a final straight. The initial straight can be used to reduce the speed before starting to turn (so as to reduce the radius of the bend that can be taken). Likewise, the final straight can be used to accelerate the car until the target velocity is reached. With addition of the extremal $\mathcal{T}$, we have an important advantage: the car can start turning while changing the speed. Likewise, when the bend finishes the car uses the $\mathcal{T}$ to anticipate the opening of the throttle and by gradually opening the bend until it turns into a straight. The resulting sequence is the following

$$\mathcal{S}\mathcal{T}_{\underline{a}}\mathcal{C}_{\underline{r}}\mathcal{S}\mathcal{C}_{\underline{r}}\mathcal{T}_{\overline{a}}\mathcal{S}.$$

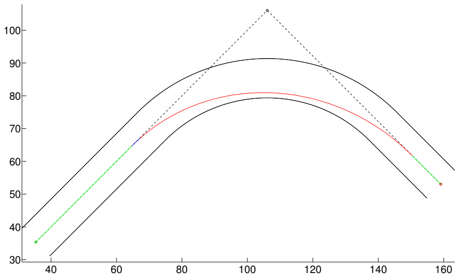

Fig. 5. An example of an optimal maneuver for a given initial and final configuration.

Similar arguments apply to a turn sector. In this case, a potentially good sequence that generalises the $\mathcal{SCS}$ sequence can be:

$$\mathcal{S}\mathcal{T}_{\underline{a}}\mathcal{C}_{\underline{r}}\mathcal{S}[\mathcal{C}_{B_r}|\mathcal{C}_{B_l}]\mathcal{S}\mathcal{C}_{\underline{r}}\mathcal{T}_{\overline{a}}\mathcal{S}.$$

In this case we make the same use of the $\mathcal{T}$ manoeuvres as in the straight line. In addition, we can have either a curve with maximum or with minimum radius $[\mathcal{C}_{B_r}|\mathcal{C}_{B_l}]$ of the sector to account for the geometry of the track (see section III-A).

Additionally, each curved maneuver has a superscript equal to $-$ or $+$ depending on the fact that the curve turns in the clockwise or counterclockwise direction (thus decreasing or increasing the value of the angle θ according to the right–hand rule).

C. Optimising Parameters

The parameters used in the optimisation for the simplified scenario are the length and the velocity at the end of the first straight line. An example of the result of the optimization for a turn sector can be observed in figure 5. The car starts and ends with equal velocities $v_i = v_e = 58.31$ m/s. It reaches 62.23 m/s at the end of the first line. Afterwards the car decelerates (shown as the blue curve) until it can accelerate to reach the correct orientation and a final velocity feasible to reach v_e with a straight line (shown as the red curve).

V. CONCLUSIONS

Local optimal planning for a robotic car racing on a track is presented. The problem of path planning has been considered first identifying the optimality of the maneuver and then giving a complete geometric analysis of the simplified scenario. In this case, sub-optimal maneuvers show to be largely dependent on the problem parameters: the weights of the time to complete the maneuver and the velocity at the end of the track as well as the dynamic properties of the vehicle completely change the maneuvers. Hence, the sub–optimal solution is given in terms of a set of functions to be minimized once the parameters of the problem are known. Future development will explicitly consider generic initial orientation and approaching maneuvers to the curve.

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