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Hybrid system reduction

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Abstract— **We propose a methodology for hybrid system model reduction that deals with the abstraction of both the continuous and the discrete behaviors of the system. Balanced residualization for continuous dynamics and pseudo-equivalent location elimination for the graph are used for model reduction in the continuous and discrete time domain respectively.**

I. Introduction

Hybrid systems (HS) are powerful models for distributed embedded systems design where discrete controls are routinely applied to continuous processes. However, the complexity of verifying and assessing general properties of HS is very high so that the use of these models is limited in applications where the size of the state space is large. To cope with complexity, abstraction and reduction are powerful techniques [1], [2], [3], [4]. Traditionally the abstraction of a system amounts to constructing an equivalent system with lower complexity. The equivalence guarantees that the results of analysis performed on the less complex system can be extended to the complex system. Several results are available in the literature for abstraction techniques in the discrete behavior (DB) (see e.g., [5]) and continuous time (CT) domain (see e.g., [6], [7]) but there are limited results for HS.

Requiring that abstractions be equivalent to the original system often yields limited reductions. Research focus is today in developing more relaxed abstraction theories for HS that can enable model simplification that goes way beyond what is possible using equivalence as a criterion to validate the reduction. An idea is to replace equivalence with approximate equivalence [8] that involves defining a metric that can quantify the distance between the system and its abstraction, and hence the quality of the abstraction.

Denoting by n_q the dimension of the continuous state x_q and $|Q|$ the number of locations of a given hybrid model H, we propose a methodology for complexity reduction that is composed by a sequence of actions that deal first with the CT components minimizing $|X|$ and then with the DB components minimizing $|Q|$. For this purpose, we combine ideas from the reduced order modeling literature (see [9]) (for the CT part) and from automata minimization methods (see [10]) (for the DB part).

The paper is organized as follows: in Section II we present basic definitions. In Section III we use the balanced residualization of the CT state $x_q(t)$ for each location q of the HS H to obtain an approximating HS H'. In Section IV the approximated equivalence definition in the labeled transition system framework is adopted to reduce the hybrid state (x,q) of H'. In Section VI, the proposed reduction technique is applied to the model of a common rail fuel injection system.

II. Preliminaries and Motivation

Hybrid systems are dynamical systems where the behavior of interest is determined by continuous and discrete dynamics interacting with each other.

Definition 1: A hybrid system (HS) is a collection $H = (Q,X,F,D,Init,E,G,R)$, where

- $Q = \{q_1, q_2, \ldots, q_{|Q|}\}$ is a finite set of discrete states or locations, where $|Q|$ is the number of locations;
- $X = \{x_1, x_2, \ldots, x_{|Q|}\}$ is a finite set of continuous states, where $x_q \in \mathbb{R}^{n_q}$ and $q \in Q$;
- $F = \{F_q = \{A_q, B_q, C_q, D_q\} | \forall q \in Q\}$ defines the continuous dynamics of the HS in each location where $A_q \in \mathbb{R}^{n_q \times n_q}$, $B_q \in \mathbb{R}^{n_q \times m_q}$ and $C_q \in \mathbb{R}^{p_q \times n_q}$, with typically $n_i \neq n_j$ for $i, j \in Q$. The continuous state $x(t)$ and the output $y(t)$ evolve in time according to the differential equations

$$\begin{array}{l} \dot{x}_q(t) = A_q x_q(t) + B_q u_q(t) \\ y_q(t) = C_q x_q(t) \end{array} ; \quad (1)$$

- $D = \{D_1, D_2, \ldots, D_{|Q|}\}$ is a set of domains or invariant conditions, where $\forall q \in Q : D_q \subseteq \mathbb{R}^{n_q}$;
- $Init \subseteq \mathbb{S} = \bigcup_{q \in Q} q \times D_q$ is the set of initial states;
- $E = \{e_{i,j} | i, j \in Q\} \subseteq Q \times Q$ is a set of edges so that $\forall i, j \in Q$ we have that $i = \varsigma(e_{i,j})$ is its source and $j = \tau(e_{i,j})$ is its target;
- $G = \{g_{i,j} \subset D_i | i, j \in Q\} \subset \mathbb{S}$ is the guard set, a subset of the hybrid state space and $\forall i, j \in Q$: $g_{i,j} \in G$, $\exists e_{i,j} \in E$. For each transition $q_i \to q_j$ the guard condition is assumed to be affine and described by

$$g_{i,j}(x_i(t)) = G^1_{i,j} x_i(t) + G^0_{i,j} = 0 \quad (2)$$

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- $R : Q \times Q \times D \to D$ is a reset map associated to each edge: with the hybrid state $s = (i,x) \in g_{i,j}$ is associated the reset function $R(j,(i,x))$ that describes the continuous states reset associated to the transitions. For each transition $q_i \to q_j$ at time t_e the reset function is assumed to be affine and described by

$$x_j(t_e^+) = R^1_{i,j} x_i(t_e^-) + R^0_{i,j} \tag{3}$$

where $R^1_{i,j} \in \mathbb{R}^{n_j \times n_i}$, $R^0_{i,j} \in \mathbb{R}^{n_j}$.

In this paper we consider HS with urgent semantics.

Given a HS H as in Definition 1, we wish to obtain a new hybrid model $H^\star$ approximately equivalent to the original one and characterized by a lower complexity. In the discrete case, complexity is associated to the number of states, while in the continuous case it is associated to the number of state variables. In the hybrid case, to reduce complexity we have to take into consideration both. It is intuitive that the continuous part is responsible for much of the complexity. To give a more convincing, albeit heuristic, argument for this fact, we discretize first the HS using a linear multi-step method such as the forward Euler integration rule. As long as the system remains in location $q \in Q$, the continuous state $x_q(t)$ evolves as defined in Eq. 1 and when a guard condition is satisfied, the system switches to a new location q' identified by the edge associated to the verified guard. At transition time t_e, the continuous state is reset as defined in Eq. 3. For each step γ during which the system is in location q, the discretized state evolution $x_q(\gamma+1) = (I + A_q)x_q(\gamma) + B_q u_q(\gamma)$ is characterized by the following complexity

$$f_q(x_q, u_q) \in \mathscr{O}(n_q^2 + n_q \cdot m_q)$$

and, as typically $n_q > m_q$, it can be approximated to $\mathscr{O}(n_q^2)$. If we assume that the HS remains in location q for a total number of Γ_q time steps during the entire integration time domain and it enters location q Λ_q times, then the complexity associated to location q is $O((\Gamma_q + \Lambda_q)n_q^2)$. Considering all the locations $|Q|$ of HS, an approximation of complexity is

$$H \in \mathscr{O}\Big(\sum_{q=1}^{|Q|} (\Gamma_q + \Lambda_q) n_q^2\Big). \tag{4}$$

From Eq. 4, the approximated complexity measure associated to the operations carried out by the numerical simulation of the HS grows linearly with the number of locations $|Q|$ while it grows quadratically with the dimension n of the continuous state. Armed with this argument, we first reduce the continuous dynamic, by selecting the continuous state partition that minimizes $|X|$ provided it is sufficiently close to the original model. Once the CT dynamics reduction has been completed, we turn to the problem of minimizing the number of locations $|Q|$.

III. Reduction of the continuous time dynamics

A. Balanced Residualized Realization

In this step, we build a reduced hybrid model H' characterized by balanced residualized continuous dynamics. Moore [11] introduced balancing for model reduction. Balancing representation consists of selecting and reducing the components of the dynamics that are "less important" in the input-output behavior of the system. The main observation is that the singular values of the controllability Gramian correspond to the amount of energy that has to be put into the system to move the corresponding states. For the observability Gramian, its singular values refer to the energy that is generated by the corresponding states.

To obtain a significative balanced reduction of the continuous dynamic in each location, it is important to consider as an output not only the effective output signal $y_q(t)$ but also the guard conditions in Eq. 2 associated to the edges e_q s.t. $q = \varsigma(e_q)$. To do that, we extend the continuous dynamic proposed in Eq. 1 as follows:

$$\begin{aligned}\dot{x}_q(t) &= A_q x_q(t) + B_q u_q(t)\\ z_q(t) &= \begin{pmatrix} y_q(t)\\ y_{q,q'_1}(t)\\ \vdots\\ y_{q,q'_m}(t)\end{pmatrix} = \begin{pmatrix} C_q\\ G^1_{q,q'_1}\\ \vdots\\ G^1_{q,q'_m}\end{pmatrix} x_q(t)\end{aligned} \tag{5}$$

where $q'_1, \ldots, q'_m \in \{i \in Q | \exists e : q = \varsigma(e),\ i = \tau(e)\}$ are all the locations reachable from q.

The transformed Gramians (see [7]) are given by $\bar{W}^c_q = T_q W^c T_q^{-1}$ and $\bar{W}^o_q = (T_q^{-1})^T_q W^o_q T_q^{-1}$ where $T_q \in \mathbb{R}^{n_q \times n_q}$ is a transformation matrix and σ^q_is are the singular Hankel values of the CT dynamic defined in Eq. 5.

Definition 2: A linear system whose Gramians are equal and have the following form

$$\bar{W}_c = \bar{W}_o = \Sigma = \mathrm{diag}\{\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n\} > 0$$

is called *balanced*.

The set $\bar{X}$ of balanced CT states is defined as

$$\bar{X} = \left\{\bar{x}_q,\ \forall q \in Q \mid \bar{x}_q = T_q x_q\right\}.$$

We can then transform the LTI system realization in each location $q \in Q$ to a balanced form by means of the transformation T_q and the balanced state vector $\bar{x}_q(t)$ can be partitioned into more significant components χ_q and less significant ones μ_q:

$$\bar{x}_q = \left(\ \chi_q \mid \mu_q\ \right)^T. \tag{6}$$

Residualization is based on the idea that the derivatives of the states μ_q corresponding to small singular Hankel values can be approximated with zero and the relative states maintained constant as long as the system remains in that location (see [12]). Reduction is then performed by selecting a particular state vector partition as in Eq. 6 according the energy of the singular Hankel values associated to the residualized components μ_q w.r.t the ones associated to the overall state x_q.

More in details, given a state vector in the balanced realization we define the following metric:

Definition 3: Let consider a balanced continuous state $x \in \mathbb{R}^n$ to which is associated the singular Hankel vector $\sigma \in \mathbb{R}^n$ and a generic state partition $i \in [1,n]$ where i specifies the number of components of χ, the distance $\psi(i) \in [0,1)$ is defined as follows:

$$\psi(i) = \begin{cases} \frac{\sum_{j=i+1}^{n} \sigma(j)}{\sum_{j=1}^{n} \sigma(j)} & \text{if} \quad i < n \\ 0 & \text{else} \end{cases}$$

Using Definition 3, given an a priori bound $\alpha \in [0,1]$ on the distance ψ, for each location $q \in Q$ consider state partition in Eq. 6 where the dimension of χ_q is given by

$$l_q = \underset{i \in [1,n_q]}{\operatorname{argmin}} \left(\psi(i) < \alpha\right). \tag{7}$$

The bound α is chosen by the designer as a trade-off between the dimension of the partitioned state χ and the quality of the system approximation. In fact, for small values of α the dimension of the state partition χ increases and for $\alpha = 0$ the partitioned state χ is equal to the original one $\bar{x}$. On the contrary, for big values of α we obtain a small dimension of the state partition χ resulting in a loose approximation.

B. Patching balanced residualized models

The result of the balanced residualization process is a set of reduced order continuous time models with possibly different state dimensions. When the overall hybrid system is considered, we need to "patch" the state equations of the various locations so that the transition from one location to another yields a consistent evolution. To do so, we have to construct an appropriate reset map for the reduced systems.

Let consider two generic locations $h,k \in Q : \exists\, e_{h,k} \in E$ so that $h = \varsigma(e_{h,k})$ is the source and $k = \tau(e_{h,k})$ is the target, with CT dynamics associated to h and k as in Eq. 5.

According to the step in Section III-A the continuous state $x_q(t)$ is transformed in a balanced form by means of the non–singular matrix T_q: $\bar{x}_q = T_q x_q$ where $q = h,k$. Applying the state reduction Eq. 7, H' is characterized by the continuous state partition presented in Eq. 6.

The system switches to the next location k at transition time t_e^- when the guard condition $g_{h,k}$ associated to the edge $e_{h,k}$ is verified, that is

$$I_{[1,l_h]\times[1,n_h]} T_h G_{hk}^1 \chi_h(t_e^-) + I_{[1,l_h]\times[1,n_h]} T_h G_{hk}^0 = 0 \tag{8}$$

where $I_{[1,l_h]\times[1,n_h]} \in \mathbb{R}^{l_h \times n_h}$ is obtained selecting the first l_h rows and n_h columns of the identity matrix $I_{n_h \times n_h}$. When the transition to location k takes place, the continuous state is mapped in the original state space of dimension n_h according the Eq. 9 computed for $t = t_e^-$:

$$\bar{x}_h(t_e^-) = \begin{cases} I_{[1,n_h]\times[1,l_h]} \chi_h(t_e^-) + I_{[1,n_h]\times[l_h+1,n_h]} \mu_h(t_e^-) & \text{if } l_h < n_h \\ \chi_h(t_e^-) & \text{if } l_h = n_h \end{cases} \tag{9}$$

where $I_{[1,n_h]\times[l_h+1,n_h]} \in \mathbb{R}^{n_h \times n_h - l_h}$ is obtained selecting the n_h rows and the last $n_h - l_h$ columns of the identity matrix $I_{n_h \times n_h}$. Then, when HS enters location k, the continuous state must be reset as follows,

$$x_k(t_e^+) = R_{hk}^1 T_h^{-1} \bar{x}_h(t_e^-) + R_{hk}^0. \tag{10}$$

IV. Approximated equivalence model reduction for the discrete component of HS

As argued in the introduction and in the previous section, insisting on strict equivalence between original and reduced HS does not yield significant simplification. We adopt here the notion of approximate equivalence. Identification techniques were the source of inspiration for using a set of metrics to evaluate the approximation quality, *distance*, between the approximate model and the complete one. In particular, two metrics are of interest:

Sum of Square Error: It is a measure of the total deviation of the fit $\hat{y}_i$ to the system response y_i (best fit $SSE = 0$): $SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$ and $SSE_{weighted} = \sum_{i=1}^{n} w_i (y_i - \hat{y}_i)^2$

R-Square. It measures how successful the fit is in explaining the variation of the data. R^2 is defined as follows:

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST} \tag{11}$$

with $SSR = \sum_{i=1}^{n} w_i (\hat{y}_i - \bar{y})^2$ and $SST = \sum_{i=1}^{n} w_i (y_i - \bar{y})^2$ R^2 assumes values lower than 1 (with 1 indicating best fit).

The technique we present here can use either measures or even others that may be found to be effective for simplification.

To reduce the discrete part of the HS model, the continuous dynamics are abstracted away introducing a labeled transition system. A *labeled transition system*, referred to as Υ, associated to a given hybrid system H is described by a formal mathematical model as follows:

Definition 4: A Labeled Transiton System (LTS) associated to the hybrid model H is a collection $\Upsilon = (Q, \Omega, E, Q^0, \Pi, \langle\langle\cdot\rangle\rangle)$, where

- $Q = \{q_1, q_2, \ldots, q_{|Q|}\}$ is the same finite set of H;
- $\Omega = \{\omega \in \{0,1\} |\, \omega = \text{label}(g(q,q'))\}$ is the set of labels or discrete inputs, where

  $$\text{label}(g(q,q')) = \begin{cases} 1 & \text{if} \quad g(q,q') \in G \quad \text{is satisfied} \\ 0 & \text{else} \end{cases}$$

  with $q, q' \in Q$. Moreover, the cardinality of the set Ω is equal to the cardinality of set of guards G.
- $E \subseteq Q \times \Sigma \times Q$ is the same finite set of H;
- $Q^0 \subseteq Q$ is the set of initial states;
- $\Pi = \{\pi_q,\, q \in Q | \pi_q \in \mathbb{R}^{n_q}\}$ is the set of observations;
- $\langle\langle\cdot\rangle\rangle$ is the observation map so that $\forall q \in Q : \pi_q = \langle\langle q \rangle\rangle$ associates to $\pi(q)$ the representation of the CT dynamics $\{\bar{A}_q, \bar{B}_q, \bar{C}_q, \bar{D}_q\}$ regulating the balanced residualized CT evolution associated to the location $q \in Q$.

Reduction is achieved using the well-known Paull-Unger recursive equivalent location definition (see [13]), where the LTS Υ can be used to identify locations to be equivalent. The significant difference here is that instead of requiring that the observations of two LTSs be and remain identical, we require that the observations of both systems be and remain close according to the metrics of choice. The distance function $d(\pi_i, \pi_j)$ computes the Euclidean norm ($\|\cdot\|_2$) of the relative difference between the representations π_i and π_j associated to the CT dynamics of locations q_i and q_j. Then the δ-equivalence between locations of a LTS is defined as follows:

Definition 5: Two states q_i and q_j belonging to the LTS Υ are approximately δ-equivalent if and only if

- the corresponding outputs $\pi(q_i)$ and $\pi(q_j)$ satisfy $d(\pi_i, \pi_j) \leq \delta$;
- for all discrete inputs ω the next states $q_i' = \tau(q_i, \omega)$ and $q_j' = \tau(q_j, \omega)$ are approximately δ-equivalent too;
- the corresponding outputs of the next states $\pi(q_i') = \langle\langle \tau(q_i, \omega)\rangle\rangle$ and $\pi(q_j') = \langle\langle \tau(q_j, \omega)\rangle\rangle$ satisfy the inequality $d(\pi_i', \pi_j') \leq \delta$;

Indistinguishability relations are computed by the *Table of Implications* Φ, obtained as the Cartesian product of the set of states by itself and illustrated in Section VI-B. Each entry of the table contains either:

- the symbol $\nsim$: if states are not equivalent;
- the symbol $\sim$: if states are approximatively equivalent (exact equivalence if $\delta = 0$);
- a pair of states (q_i', q_j') whose equivalence implies the equivalence of the states (q_i, q_j) in the entry.

By solving the implications defined in the third type of entries, the *Table of Implications* is refined until all entries are assigned to either $\nsim$ or $\sim$. The refined table defines the equivalent pairs of locations and hence a minimal LTS $\tilde{\Upsilon}_\delta$ that is δ-equivalent to Υ, or $\tilde{\Upsilon}$ if all the equivalence relations are satisfied for $\delta = 0$.

Once the *Table of Implications* Φ is completely solved for the hybrid model H', we can reduce the cardinality of the set Q' of H' removing the equivalent locations and obtaining the new minimal set $Q^\star$.

V. Discrete state abstraction for HS with a given structure

HS have often a particular structure that depends on the application considered. Exploiting this structure, a HS can be further simplified.

Consider a HS consisting of two subsystems in cascade or feedback configuration [14]. The composition of these two systems is a unique hybrid model characterized by a finite set Q due to the Cartesian product of the sets Q_l of the l hybrid models involved in the composition. Reducing the two subsystems before performing the composition can result in significant savings. To do so we can certainly apply the techniques described above. Further, if the discrete part of the hybrid subsystems is characterized by simple cycle that exhibits fast-switching behavior, a potential reduction is to replace the discrete states with a single "mean-value" state that eliminates completely the discrete part of the hybrid subsystem reducing it to a pure continuous time system. Accepting this reduction depends on the metrics of choice as presented in the previous section.

A model of the mean-value CT behavior of the system performing the cycle at high frequency can be obtained with variable–structure control theory in case the system switches across a sliding surface (see [15], [16], [17]). Alternatively, the model can be obtained by identification on simulation traces of the HS.

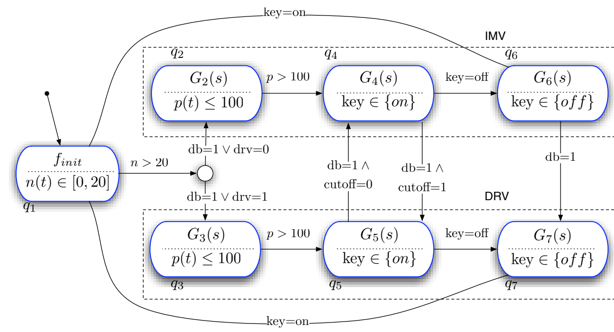

Fig. 1. Hybrid model of the combustion quality controller.

VI. Example: common rail fuel injection system

As an example of application of the proposed methodology, consider the common rail fuel injection model H_{CR} presented in [18]. The hybrid model H_{CR} is obtained by the composition of several hybrid models, among which the HP pump model and the injector model have discrete behaviors described by independent cyclic paths. More in details, the HP pump is composed by three volumetric rams, each of them is modeled by three locations to reproduce the intake phase and by two locations for the compression one. Considering the injector, its open and closed conditions are modeled by two locations and, as in a multi-jet Diesel engine there are typically three injections for each stroke, we obtain a hybrid model with six locations for each injector. Moreover, considering the overall fuel injection system architecture, it contains two feedback loops where outer one is composed by the Combustion Quality Controller (*CQC*), which defines the rail pressure reference based on the engine conditions and the requested engine torque. These references represent the input signals for the inner common rail model controlled by the rail pressure controller.

The structure of the (*CQC*) is shown in Figure 1, where locations are related to the different engine operative modes (i.e. cranking, cut-off and power-off) and different actuators to be used for rail pressure control: either the IMV (a valve mounted on the HP pump) or the DRV (a valve mounted on the rail). Notice that *CQC* is designed to work properly in engine equipped with IMV only, DRV only, or both. In the last case, the *CQC* selects on–line the type of actuator to be used to minimize fuel consumption, engine noise and tailpipe emissions. The initial location is q_1 that represents the starting engine operative mode. After cranking, the *CQC* switches to the next location, which depends on the actuator configuration: db $= 1$ denotes that both IMV and DRV are present. In locations q_2 and q_3, the common rail model acts open–loop, until the fuel pressure reaches 100 bar. Then, *CQC* moves to closed–loop operation modes, either to q_4 or q_5 depending on the chosen actuation. During fuel cut-off, *CQC* always switches to q_5, if DRV is present. For a proper power–off of the engine, *CQC* switches either to location q_6 or to location q_7.

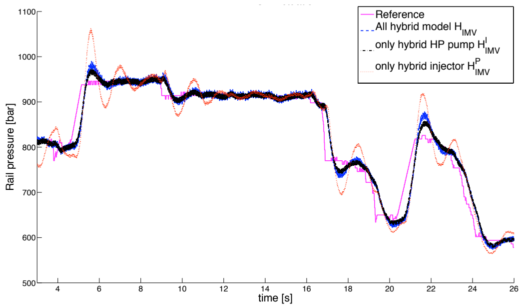

Fig. 2. Profiles of the rail pressure evolution obtained with the abstracted models H_{CR}^{HP} and H_{CR}^{INJ} and the original model H_{CR}.

A. *Discrete dynamics abstraction of the common rail components*

In this subsection, we apply the *discrete dynamics abstraction* step to the hybrid models describing the injectors and the HP pump to reduce the number of locations of the common rail model obtained by the composition of these components. Hence, according to the step in V, independent cyclic paths are identified and the corresponding simplifications of H_{CR} can be investigated to verify whether they are accurate enough for rail pressure controller design. Step in V is performed by computing mean–value signals in time obtaining the following two abstracted models:

- H_{CR}^{HP}: obtained by replacing in H_{CR} the HP pump hybrid model with: $Q_{out}^{HP}(s) = \eta \frac{1}{1+\tau s} Q_{in}^{HP}(s)$ where η is HP pump efficiency, and Q_{out}^{HP} and Q_{in}^{HP} are respectively the output and input HP pump flow rates;
- H_{CR}^{INJ}: obtained by replacing in H_{CR} the injector hybrid model with: $Q_{out}^{INJ}(t) = \sum_{i=\{pil,pre,main\}} \frac{n}{30} f_{inj}(ET_i)$ where f_{inj} is a piecewise affine function modeling injector flow rate, ET_i is the injection time and n is the engine speed revolution.

In model H_{CR}^{HP} the HP pump is considered as a continuous system that provides the rail with the mean–value flow of the original hybrid HP pump model.

In model H_{CR}^{INJ}, the discrete behavior of fuel injections is abstracted away and injectors are modeled as continuous valves that deliver mean–value fuel flow to the combustion chamber.

According the metric illustrated in IV, we verify whether the abstractions H_{CR}^{HP} and H_{CR}^{INJ} of H_{CR} are accurate enough for controller design (see [18]). Simulation results in Figure 2 show that while model H_{CR}^{INJ} with abstracted injector reproduces closely the evolution of the original model H_{CR}, the behavior of the model H_{CR}^{HP} with pump abstraction is not satisfactory. Model H_{CR}^{HP} correctly reproduces the steady–state behavior but not the transient one. In addition, model H_{CR}^{HP} does not exhibit the high frequency rail pressure ripple that is present in the original model H_{CR}, while the ripple is precisely reproduced by model H_{CR}^{INJ}. This qualitative analysis illustrated in [18] is confirmed by the computation of metrics. Index R^2 in Eq. 11 evaluates to 96,95% for H_{CR}^{INJ}, and 69,46% for H_{CR}^{HP}. Since a standard threshold for good model fitting using R^2 is 90%, then H_{CR}^{INJ} is a good abstraction while H_{CR}^{HP} is not. Therefore, for rail pressure controller design, the discrete behavior of the injectors can be approximated away, modeling fuel injections as a continuous phenomenon, while the discrete behavior of the HP pump has to be represented because it affects significantly the closed–loop behavior.

B. *Hybrid behavior reduction of the fuel injection system*

We illustrate in this subsection the application of the proposed methodology to the fuel injection system shown in Figure 1. According the Section III, the first step of the reduction of the CT dynamic is the projection of the dynamic state in a reduced state space and successively keeping only the dominant components. In particular, a linear transfer function obtained by the composition of the components in the common rail model and rail pressure controller is associated to each location of the *CQC*. More in details, to location q_4 is associated a 3-rd order CT dynamic described by the transfer function in Eq. 12

$$\frac{P_{rail}(s)}{P_{ref}(s)} = G_4(s) = \frac{51.48s + 52.1}{s^3 + 13.78s^2 + 55.46s + 52.42}. \quad (12)$$

Applying the balancing transformation $T_q \in \mathbb{R}^{3\times 3}$ we obtain the following singular Hankel values vector $\sigma_{q_4} \in \mathbb{R}^3$: $\sigma_{q_4} = \begin{pmatrix} 0.7085 & 0.1577 & 0.0538 \end{pmatrix}$ and choosing $\alpha = 0.1$ a reduced 2-nd order dynamical model $\tilde{G}_4(s)$ is obtained. This value of α guarantees a proper reproduction of the inner loop dynamic behavior. In fact the metric index R^2 between the 3-rd and 2-nd order models is equal to 93%, while for a bigger value of α, the reduced model does not guarantee an accurate performances.

To the location q_5 the 3-rd order system expressed in Eq. 13 is associated:

$$\frac{P_{rail}(s)}{P_{ref}(s)} = G_5(s) = \frac{50.24s + 50.74}{s^3 + 13.62s^2 + 54.26s + 51}. \quad (13)$$

and then the corresponding singular Hankel values vector $\sigma_{q_5} \in \mathbb{R}^3$ is $\sigma_{q_5} = \begin{pmatrix} 0.7083 & 0.1575 & 0.0533 \end{pmatrix}$.

By imposing the same value of $\alpha = 0.1$, the resulting 2-nd order model $\tilde{G}_5(s)$ guarantees a good approximation of the original one: in this case the metric index R^2 between the 3-rd and 2-nd order models is equal to 94%.

When considering locations q_2 and q_3, the associated models are characterized by the same dynamic parameters as in these locations the system acts in open loop mode and the rail pressure controller maintains the IMV valve open and/or the DRV valve closed. In these locations, the rail pressure evolution depends only on the geometry of the mechanical component and a 1-st order model (see Eq.s 14) has been associated.

$$\frac{P_{rail}(s)}{P_{ref}(s)} = G_2(s) = G_3(s) = \frac{0.7531}{s + 0.7537}. \quad (14)$$

A similar result holds for the pair of locations q_6 and q_7 representing the engine power off mode. In this operative situation, for IMV implementation, the rail pressure decreases by means of a particular actuation of the injectors, while for

DRV implementation, the rail pressure decreasing is due to the DRV flow rate. For any kind of implementation, the same first order transfer function reported in Eq. 15 is obtained:

$$\frac{P_{rail}(s)}{P_{ref}(s)} = G_6(s) = G_7(s) = \frac{0.006433}{s + 0.1123}. \qquad (15)$$

We can now apply the *approximate equivalence reduction*, and associating a LTS to the hybrid model we obtain the following Table of Implications.

q_2	$\nsim$					
q_3	$\nsim$	(q_4, q_5)				
q_4	$\nsim$	$\nsim$	$\nsim$			
q_5	$\nsim$	$\nsim$	$\nsim$	(q_6, q_7)		
q_6	$\nsim$	$\nsim$	$\nsim$	$\nsim$	$\nsim$	
q_7	$\nsim$	$\nsim$	$\nsim$	$\nsim$	$\nsim$	$\sim$
	q_1	q_2	q_3	q_4	q_5	q_6

Applying the Definition 5 to the pair of locations q_6 and q_7, it follows from Eq. 15 that $d(\pi_6, \pi_7) = 0$ and as a consequence we have an *exact* equivalence relation between these locations. The equivalence between the pair (q_6, q_7) implies an equivalence relation between locations $q4$ and q_5 where, due to Eq. 12 and 13 the maximum relative difference $d(\pi_4, \pi_5)$ is equal to 2.37%. Hence these locations are approximatively equivalent implying also the equivalence between the pair (q_2, q_3). As the same continuous transfer function is associated to locations q_2 and q_3 (see Eq. 14), the equivalence conditions are satisfied for $\delta = 0$ implying an exact equivalence.

As a consequence, it is possible to define the new set of locations $Q^{\star} = \{q_1, \; q_2, \; q_4, \; q_6\}$ characterized by a lower cardinality $\bar{Q}$ (4 locations) than the original one (7 locations). As in the fuel injection model in Figure 1 there are not fast–switching cyclic paths, the original hybrid model in Figure 1 can be reduced to $H^{\star}$ illustrated in Figure 3.

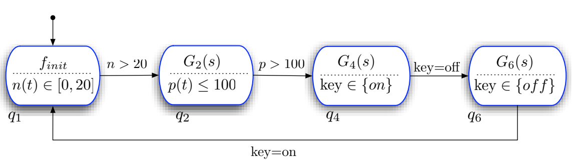

Fig. 3. Reduced model of the injection system.

VII. Conclusion

We presented a methodology for the construction of a reduced order hybrid model $H^{\star}$ that is approximatively equivalent to a given hybrid system H. To deal with the continuous and discrete dynamics, balanced residualization theory and discrete locations pseudo-equivalence were used. A definition of distance between the original hybrid model H and an approximatively equivalent one was proposed. The effective reduction of the computational cost of the hybrid model of the common rail fuel injection system was presented.

The approach proposed here can be considered as a first step towards a theory and a set of algorithms for reduced-order modeling of hybrid systems. There are several numerically attractive version of the Truncated Balanced Reduction family that make use of the Krylov subspace methods. These methods obtain moment matching implicitly using only matrix multiplications and are very effective in making the very expensive TBR methods usable. Along this line, we plan to extend these methods to the case of Linear Time Varying systems as well as non linear CT systems where new results surfaced in the recent past that make us believe we can reduce significantly even these more general hybrid systems. To evaluate the sensitivity of the discrete system behavior w.r.t the CT approximated subsystems, stochastic perturbations will be applied to the CT signals to evaluate the effects on the signals generated by the discrete components.

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