ADAPTIVE NONLINEAR CONTROL OF DYNAMIC MOBILE ROBOTS WITH PARAMETER UNCERTAINTIES

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Abstract: Research of a modular stabilizing control law for uncertain, nonholonomic mobile systems with actuators limitation has been investigated. Modular design allows the definition of a stabilizing control law for the kinematic model. The presence of uncertainties in the actuators parameters or in the vehicle dynamics has been treated both adding suitable components to the Lyapunov function and using parameters adaptation laws (e.g. adaptive control and backstepping techniques). Simulations are reported for the set point stabilization of a unicycle like vehicle showing the feasibility of the proposed approach. Torque limitations for a unicycle like vehicle has been investigated using backstepping techniques for the vehicle tracking problem. Simulations are reported.

Keywords: Nonholonomic systems, adaptive control, limited actuation, vehicle control

1. INTRODUCTION

Practical control of robots involves both the design of state feedback control laws and the modelling of the physical robotic platform used. Notwithstanding the efforts spent to identify the parameters of the mechanical system, control law robustness is more often increased using adaptive control (Jiang *et al.*, 2004). On the other hand, mechanical systems suffer also of limits on actuation that should be taken into account to prevent instability. Combining parameters uncertainties with actuator limits carries to a challenging yet open problem, particularly if the system to control is nonholonomic.

The problem of actuator limitations for velocity control of kinematic nonholonomic systems has been solved for example in (Sontag and Malisoff, 1999) defining universal formulas for asymptotic stability, in (Nijmeijer *et al.*, 2001), with time varying control Lyapunov functions, or in (Beard and Ren, 2004) for air vehicles' control. In the present paper, the problem of adaptive nonlinear control for generic kinematic nonholonomic systems in the presence of actuator limits is considered assuming a linear relation between the control space and the uncertainties.

In literature, the problem of stabilizing a unicycle like vehicle (i.e. driftless nonholonomic system)

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with both uncertain actuators and dynamic parameters has been solved for regulation (Aguiar *et al.*, 2000) and for path–following (Soetano *et al.*, 2003) using adaptive nonlinear and backstepping control and assuming the knowledge of the uncertainty sign. In (Tso *et al.*, 2000) the tracking control of uncertain dynamic nonholonomic system has been solved for systems transformable in *Extended One–Generator Multi–Chain Form*, defining universal formulas for Lyapunov stabilization.

To the best of authors' knowledge, while the tracking problem has been solved considering the presence of maximum velocity constraints, very little has been done in the investigation of tracking control laws with limited torques.

This paper presents an attempt to solve the control of nonholonomic mobile robots using adaptive and switching control in the presence of uncertainties related to actuation and dynamic, coping with limitations on actuator saturations. The underlying idea is that each component of the final control law can be modularly composed using Lyapunov functions, starting from a stabilizing controller thought for the kinematic model. The problem of torque limitations for the unicycle tracking problem has been solved using switching techniques. Simulations are reported demonstrating the feasibility of the proposed approach.

2. ADAPTIVE NONLINEAR CONTROL

Let us consider a generic, nonholonomic, input–affine nonlinear system

$$\dot{q} = f(q) + g(q)u \tag{1}$$

where $q \in \mathbb{R}^n$ is the state space vector, $f(q)$ and $g(q)$ are the drift and the input vector fields respectively and $u \in \mathbb{R}^m$ are the available controls. Let $u(q) \in U \subset \mathbb{R}^m$ be the control law that asymptotically stabilizes the kinematic system. Indeed, it exists a positive definite Lyapunov function $\mathbf{V}_1(q) > 0$, with

$$\dot{\mathbf{V}}_1(q) = \nabla\mathbf{V}_1(q)\left[f(q) + g(q)u(q)\right] < 0 \tag{2}$$

whenever $q \neq 0$.

Consider now a new input space $V \subset \mathbb{R}^m$ and the isomorphism $\tilde{F}^{-1} : U \rightarrow V$. The control input $u_\nu \in V$ may be, for instance, the actual, low level velocity vector available on the physical nonholonomic system, while the control input u can be viewed as a control abstraction, e.g. the steering velocities of the kinematic system model. The stabilizing controller will be trivially $u_\nu = \tilde{F}^{-1}(u)$. Let the isomorphism be a bilinear w.r.t. U and some parameters $\eta \in \mathbb{R}^p$, then:

$$u_\nu = \tilde{F}^{-1}(u, \eta) = F^{-1}(u)\Theta_\eta \bar{\mathbf{1}}$$

where, assuming $p = m$, $\bar{\mathbf{1}} = [1, 1, \ldots, 1]^T \in \mathbb{R}^p$ and $\Theta_\eta = \mathrm{diag}(\eta_i)$. From the hypothesis of the isomorphism $\tilde{F}^{-1}$, it is possible to define the invertible mapping function and the relative Lyapunov function:

$$\begin{array}{l} u = \tilde{F}(u_\nu, \eta) = F(u_\nu)\Theta_\eta^{-1}\bar{\mathbf{1}} \\ \dot{\mathbf{V}}_1(q, \eta) = \nabla\mathbf{V}_1(q)\left[f(q) + g(q)F(u_\nu)\Theta_\eta^{-1}\bar{\mathbf{1}}\right] < 0 \end{array}$$

Considering an imperfect knowledge of the parameters η, $\dot{\mathbf{V}}_1(q, \eta)$ is not defined any further. Defining $\tilde{\eta} = \hat{\eta} - \eta$ as the parameters error on the estimations $\hat{\eta}$, we have:

$$\begin{array}{l} \hat{u}_\nu(q) = \tilde{F}^{-1}(u, \hat{\eta}) = F^{-1}(u)\hat{\Theta}_\eta\bar{\mathbf{1}} \\ \hat{u}(q) = \tilde{F}(\hat{u}_\nu, \eta) = F(\hat{u}_\nu)\Theta_\eta^{-1}\bar{\mathbf{1}} \\ u(q) = \tilde{F}(u_\nu, \eta) = \tilde{F}(\hat{u}_\nu, \hat{\eta}) = F(\hat{u}_\nu)\hat{\Theta}_\eta^{-1}\bar{\mathbf{1}} \end{array}$$

where $u(q)$ is the desired, kinematically stabilizing control law, $\hat{u}(q) \in U$ and $\hat{u}_\nu(q) \in V$ are the actual control input and the low level control input vectors respectively, affected by the parameter estimate $\hat{\eta}$. The control $\hat{u}(q)$ is applied to the system that has the true parameter η. Using the control Lyapunov function $\mathbf{V}(q) = \mathbf{V}_1(q)$, the time derivative becomes:

$$\begin{aligned} \dot{\mathbf{V}}(q, \eta, \tilde{\eta}) &= \nabla\mathbf{V}_1(q)[f(q) + g(q)F(\hat{u}_\nu)\Theta_\eta^{-1}\bar{\mathbf{1}}] \\ &= \nabla\mathbf{V}_1(q)[f(q) + g(q)U_q\hat{\Theta}_\eta\Theta_\eta^{-1}\bar{\mathbf{1}}] \\ &= \nabla\mathbf{V}_1(q)f(q) + \\ &\quad \nabla\mathbf{V}_1(q)g(q)U_q\left(\tilde{\Theta}_\eta + \Theta_\eta\right)\Theta_\eta^{-1}\bar{\mathbf{1}} \\ &= \dot{\mathbf{V}}_1(q) + \nabla\mathbf{V}_1(q)g(q)U_q\tilde{\Theta}_\eta\Theta_\eta^{-1}\bar{\mathbf{1}} \end{aligned}$$

where $U_q = \mathrm{diag}(u_i(q))$.

Under the assumption that the parameters are unknown but constant, i.e. $\dot{\tilde{\eta}} = \dot{\hat{\eta}}$, consider $\mathbf{V}(q, \eta, \tilde{\eta}) = \mathbf{V}_1(q) + \mathbf{V}_\eta(\eta, \tilde{\eta})$, where:

$$\begin{cases} \mathbf{V}_\eta(\eta, \tilde{\eta}) = \dfrac{1}{2}\bar{\mathbf{1}}^T\tilde{\Theta}_\eta^T\Theta_\eta^{-T}\Gamma\tilde{\Theta}_\eta\bar{\mathbf{1}} > 0 \\ \dot{\mathbf{V}}_\eta(\eta, \tilde{\eta}) = \bar{\mathbf{1}}^T\tilde{\Theta}_\eta^T\Theta_\eta^{-T}\Gamma\dot{\tilde{\Theta}}_\eta\bar{\mathbf{1}} \end{cases} \tag{3}$$

where $\Gamma > 0$, symmetric and depends on the sign of the uncertainties to verify $\mathbf{V}_\eta(\eta, \tilde{\eta}) > 0$ (the sign assumption can be found also in (Soetano *et al.*, 2003) and (Aguiar *et al.*, 2000)).

Choosing the adaptation of the uncertain parameters as:

$$\dot{\tilde{\Theta}}_\eta\bar{\mathbf{1}} = \dot{\tilde{\eta}} = \dot{\hat{\eta}} = -\Gamma^{-1}U_q^T g(q)^T\nabla\mathbf{V}_1(q)^T, \tag{4}$$

the time derivative of the Lyapunov function (3) becomes [3]:

$$\dot{\mathbf{V}}_\eta(q, \eta, \tilde{\eta}) = -\nabla\mathbf{V}_1(q)g(q)U_q\tilde{\Theta}_\eta\Theta_\eta^{-1}\bar{\mathbf{1}}$$

that ensures the perfect compensation of the undefined sign term. Hence, the final control Lyapunov function and its time derivative are:

$$\begin{cases} \mathbf{V}(q, \eta, \tilde{\eta}) = \mathbf{V}_1(q) + \mathbf{V}_\eta(\eta, \tilde{\eta}) > 0 \\ \dot{\mathbf{V}}(q, \eta, \tilde{\eta}) = \dot{\mathbf{V}}_1(q) < 0 \end{cases},$$

[3] where we use the fact that the diagonal matrices $\tilde{\Theta}_\eta$ and Θ_η^{-1} commute.

and the system with uncertain parameters inherits the stability features (simple or asymptotic) of the kinematic system without uncertain parameters since $\dot{\mathbf{V}}(q,\eta,\tilde{\eta})$ is negative semidefinite with respect to the whole state space $(q,\tilde{\eta})=(0,\tilde{\eta})$. The uncertain parameter estimation does not necessarily converges to zero, as is usual in the adaptive control framework, while the system correctly does its job (this can be proved using LaSalle's theorem (Hahn, 1963)).

2.1 Actuator limits

Let us consider $\eta_i > 0$, with $i = 1,\ldots,m$ and $\dot{\eta}_i = 0$, and a limited input velocity $|\hat{u}_i| \le u_{\max_i}$, with $u_{\max_i} > 0, \forall i = 1,\ldots,m$. The velocity constraint is satisfied if

$$|\hat{u}(q)| = |\tilde{u}(q) + u(q)| \le |\tilde{u}(q)| + |u(q)| \le u_{\max} .$$

Due to the presence of uncertainties, the velocity error in the low level inputs $\tilde{u}_\nu(q) = \hat{u}_\nu(q) - u_\nu(q)$ can be rewritten as

$$\tilde{u}_\nu = F^{-1}(u(q))\tilde{\Theta}_\eta \bar{\mathbf{1}} \qquad (5)$$

that ensures the separation of each uncertainty with respect to the control input space (for the linearity of the input transformation inverse $\tilde{F}^{-1}$). Applying the linear operator F to (5) and multiplying both side by $\Theta_\eta^{-1}\bar{\mathbf{1}}$, one gets:

$$\tilde{u} = F(\tilde{u}_\nu(q))\Theta_\eta^{-1}\bar{\mathbf{1}} = U_q\tilde{\Theta}_\eta\Theta_\eta^{-1}\bar{\mathbf{1}} \qquad (6)$$

For simplicity's sake, let us now examine each single uncertainty separately, since the parameter matrices Θ are of diagonal form. Consider

$$|\tilde{u}_i(q)| = |u_i(q))| \left|\frac{\tilde{\eta}_i}{\eta_i}\right| .$$

The limited velocity constraint affects the desired control inputs $u_i(q)$, with $i = 1,\ldots,m$, the uncertainty parameters error and the true value:

$$|\hat{u}_i(q)| \le |\tilde{u}_i(q)| + |u_i(q)| = |u_i(q)| \left(\left|\frac{\tilde{\eta}_i}{\eta_i}\right| + 1\right) .$$

Unfortunately, no assumptions can be made on the values of the estimation parameters along the controlled trajectories of the system. However, if

$$\left|\frac{\tilde{\eta}_i}{\eta_i}\right| \le k-1, \ \forall i = 1,\ldots,m, \text{ and } k > 1 \qquad (7)$$

holds for each time $t > t_0$, where t_0 is the starting time, the limited velocity constraint is imposed directly on the desired, perfectly known, kinematic control $u(q)$, since

$$|\hat{u}_i(q)| \le |\tilde{u}_i(q)| + |u_i(q)| \le k|u_i(q)| \Rightarrow$$
$$|u_i(q)| \le \frac{1}{k}u_{\max_i}, \ \forall i = 1,\ldots,m$$

Since $\eta_i > 0$, by letting the uncertain parameters $0 \le \tilde{\eta}_i \le (k-1)\eta_i$, the condition (7) is fulfilled in the initial configuration (i.e. at t_0). Nevertheless, the constraints on the estimated parameters have to be satisfied on all the possible trajectories of the nonholonomic system. Choosing the Lyapunov function (3) and the adaptation parameters law (4), the nonholonomic system is controlled with limited controls if it is possible to properly tuning the estimation parameter weighting matrix Γ using LaSalle's theorem. Intuitively, if $\|\Gamma\| \to +\infty$, $\mathbf{V}(q,\eta,\tilde{\eta}) \approx \mathbf{V}_\eta(\eta,\tilde{\eta})$, by using $\dot{\mathbf{V}}(q,\eta,\tilde{\eta}) \le 0$ and $\dot{\eta} = 0$, with $\eta_i > 0$, it is possible to assert that $\|\tilde{\Theta}_\eta\|$ will be not increasing as $t \to +\infty$. This limit condition is enforced by the orthogonality of q and η and by the fact that the two component Lyapunov functions $\mathbf{V}_1(q)$ and $\mathbf{V}_\eta(\eta,\tilde{\eta})$ depend on q and η separately.

3. ADAPTIVE CONTROL FOR DYNAMIC UNCERTAINTIES

Let us consider a general mechanical system with nonholonomic constraints:

$$\begin{aligned} &B^\star(q)\ddot{q} + C^\star(\dot{q},q)\dot{q} + G_u^\star(q) + A(q)^T\lambda = W^\star(q)\tau \\ &A(q)\dot{q} = 0 \Rightarrow \dot{A}(q)\dot{q} + A(q)\ddot{q} = 0 \end{aligned} \qquad (8)$$

Using standard manipulations of constrained dynamic systems (see for example (Tso *et al.*, 2000)), it is possible to decompose the system into two parts: the kinematic model and the relative dynamic model. Hence, let us consider a generic dynamic nonholonomic system, with drift term:

$$\begin{cases} \dot{q} &= f(q) + g(q)u \\ \dot{u} &= B(q,\eta)^{-1}(W(q)\tau - C(\dot{q},q,\eta)u - G(q,\eta) + \\ & \quad +\gamma^\star(q,\eta)) \end{cases} \qquad (9)$$

where $q \in \mathbb{R}^n$ is the state vector of the kinematic model state space (i.e. generalized system variables), $f(q)$ and $g(q)$ are the system vector fields and $u \in \mathbb{R}^m$ are the kinematic controls [4]. $\eta \in \mathbb{R}^p$ are the dynamic parameters of the mechanical nonholonomic system. Furthermore, $\gamma^\star(q,\eta)$ is a generic non linear term that is supposed to be linear with respect to the dynamic parameter $\gamma^\star(q,\eta) = \gamma(q)\eta$, that could appear from changes of coordinates. It is straightforward that:

$$B(q,\eta)\dot{u} + C(\dot{q},q,\eta)u + G(q,\eta) = Y(\dot{u},u,\dot{q},q)\eta$$

where $Y(\dot{u},u,\dot{q},q)$ is the well known matrix regressor. Suppose that a stabilizing control law u for the kinematic system exists. Hence, there exists a positive definite Lyapunov function $\mathbf{V}_1(q)$ whose time derivative satisfies (2). The same kinematic control law can be used also with the full dynamic system as the virtual control of a backstepping problem. Define $\tilde{u}(q) = u_\tau(q) - u(q)$

[4] The generic dynamic matrix $M^\star$ in (8) changes to M in (9) to highlight the nonholonomic constrained dynamic.

as the control error, where u_τ is the dynamic system variable, i.e. the velocity control law of the kinematic subsystem. Consider the torque control law:

$$\tau = W(q)^{-1}(B(q,\eta)\dot{u}(q) + C(\dot{q},q,\eta)u(q) + \\ +G(q,\eta) - \gamma^\star(q,\eta) - K_b\tilde{u} - \Delta) \tag{10}$$

with K_b a square, positive definite matrix (the backstepping gain) that gives:

$$B(q,\eta)\dot{\tilde{u}} = -K_b\tilde{u} - \Delta - C(\dot{q},q,\eta)\tilde{u}$$

and $\Delta = (\nabla\mathbf{V}_1(q)g(q))^T$.

To prove that the proposed control law effectively stabilizes the system, we use the following control Lyapunov function:

$$\mathbf{V}_2(q,\tilde{u}) = \mathbf{V}_1(q) + \frac{1}{2}\tilde{u}^T B(q,\eta)\tilde{u}, \tag{11}$$

having time derivative:

$$\begin{aligned}\dot{\mathbf{V}}_2(q,\tilde{u}) &= \nabla\mathbf{V}_1(q)f(q) + \Delta^T u_\tau + \tilde{u}^T B(q,\eta)\dot{\tilde{u}} + \\ &\quad +\frac{1}{2}\tilde{u}^T\dot{B}(q,\eta)\tilde{u} \\ &= \nabla\mathbf{V}_1(q)f(q) + \Delta^T u - \tilde{u}^T K_b\tilde{u} \\ &\quad +\frac{1}{2}\tilde{u}^T\left(\dot{B}(q,\eta) - 2C(\dot{q},q,\eta)\right)\tilde{u} \\ &= \dot{\mathbf{V}}_1(q) - \tilde{u}^T K_b\tilde{u}\end{aligned} \tag{12}$$

that is clearly negative definite. It is worth noting that the term added in (11) represents the kinetic energy of the vehicle and that $\dot{B}(q,\eta) - 2C(\dot{q},q,\eta)$ (that is skew-symmetric) represents the Hamilton's principle on the energy conservation [5]. The first term of the right side of (12) ensures the stability of the system while the second term ensures the convergence of $\tilde{u} \to 0$. As the backstepping gain matrix K_b increases, the latter convergence velocity increases as it is increasing the control effort as well.

For ease of notation, in what follows, we will suppress the explicit dependence of system matrices and controls by $q, \dot{q}, u, \dot{u}$. Consider now a partial knowledge of the dynamic parameters $\hat{\eta}$. Let $\tilde{\eta} = \hat{\eta} - \eta$ be the parameter estimation error and with $\tilde{M}(\tilde{\eta}) = \hat{M}(\hat{\eta}) - M(\eta)$ the estimation error on the generic system matrix M due to parameter uncertainties.

The torque control law is then:

$$\tau = W^{-1}(\hat{B}(\hat{\eta})\dot{u} + \hat{C}(\hat{\eta})u + \hat{G}(\hat{\eta}) + \\ -\hat{\gamma}^\star(\hat{\eta}) - K_b\tilde{u} - \Delta), \tag{13}$$

that replaced in $B(\eta)\dot{\tilde{u}}$ gives:

$$B(\eta)\dot{\tilde{u}} = -K_b\tilde{u} - \Delta + Y\tilde{\eta} - \gamma\tilde{\eta} - C(\eta)\tilde{u} \tag{14}$$

(recall that $\gamma^\star$, $\tilde{\gamma}^\star$, $\hat{\gamma}^\star$ are linear w.r.t. η, $\tilde{\eta}$ and $\hat{\eta}$ respectively).

[5] The skew-symmetric property holds for a particular definition of $C(\dot{q},q,\eta)$.

Let $\dot{\eta} = 0$, i.e. constant unknown dynamic parameters, and the adaptation law of the parameters:

$$\dot{\tilde{\eta}} = \dot{\hat{\eta}} = -\Gamma^{-1}(Y^T - \gamma^T)\tilde{u} \tag{15}$$

and consider the composite Lyapunov function, with its time derivative:

$$\begin{cases}\mathbf{V}_3(q,\tilde{u},\tilde{\eta}) = \mathbf{V}_2(q,\tilde{u}) + \frac{1}{2}\tilde{\eta}^T\Gamma\tilde{\eta} \\ \dot{\mathbf{V}}_3(q,\tilde{u},\tilde{\eta}) = \nabla\mathbf{V}_1(q)f + \Delta^T u_\tau + \tilde{u}^T B(\eta)\dot{\tilde{u}} + \\ \qquad +\frac{1}{2}\tilde{u}^T\dot{B}(\eta)\tilde{u} + \tilde{\eta}^T\Gamma\dot{\tilde{\eta}}\end{cases} \tag{16}$$

where $\Gamma > 0$ and symmetric (idependent from the dynamic parameters sign).

Replacing (14) and (15) in (16), we obtain:

$$\begin{aligned}\dot{\mathbf{V}}_3(q,\tilde{u},\tilde{\eta}) &= \nabla\mathbf{V}_1(q)f + \Delta^T u - \tilde{u}^T K_b\tilde{u} \\ &= \dot{\mathbf{V}}_2(q,\tilde{u})\end{aligned} \tag{17}$$

that is, once again, negative semidefinite if the desired kinematic control law makes $\mathbf{V}_1(q)$ negative definite. Therefore, the native control law $u(q)$ is used to stabilize the nonholonomic system since the kinematic control error $\tilde{u} \to 0$; the parameter estimation $\hat{\eta}$ does not converge necessarily to η, but still allows for the control task to be solved.

4. ADAPTIVE CONTROL FOR DYNAMIC AND ACTUATOR UNCERTAINTIES

Consider again the mechanical system (8) and the stabilizing law (10). Let $\tau_\nu \in \mathbb{R}^m$ be a different set of torque inputs, related to some actuators' parameters $\eta_a \in \mathbb{R}^p$, whose generic non linear relation is $\tau = \tilde{F}_\tau(\tau_\nu, \eta_a)$. The full nonholonomic dynamics become:

$$\begin{cases}\dot{q} = f + gu \\ \dot{u} = B(\eta)^{-1}(W\tilde{F}_\tau(\tau_\nu,\eta_a) - C(\eta_d)u - G(\eta_d) + \\ \qquad +\gamma^\star(\eta_d))\end{cases}$$

where η_d are dynamic parameters. The stabilizing control law for the new set of inputs is clearly: $\tau_\nu = \tilde{F}_\tau^{-1}(\tau,\eta_a)$. Defining $\tilde{\eta}_a = \hat{\eta}_a - \eta_a$ as the parameters error, function of the parameter estimation $\hat{\eta}_a$, and supposing that the new input field $\tilde{F}_\tau$ is bilinear w.r.t. τ and η_a, it is possible to assert that:

$$\begin{aligned}\hat{\tau}_\nu &= \tilde{F}_\tau^{-1}(\tau,\hat{\eta}_a) = F_\tau^{-1}(\tau)\hat{\Theta}_{\eta_a}\bar{\mathbf{1}} \\ \hat{\tau} &= \tilde{F}_\tau(\hat{\tau}_\nu,\eta_a) = F_\tau(\hat{\tau}_\nu)\Theta_{\eta_a}^{-1}\bar{\mathbf{1}} \\ \tau &= \tilde{F}_\tau(\tau_\nu,\eta_a) = \tilde{F}_\tau(\hat{\tau}_\nu,\hat{\eta}_a) = F_\tau(\hat{\tau}_\nu)\hat{\Theta}_{\eta_a}^{-1}\bar{\mathbf{1}}\end{aligned}$$

It is worth noting that $\hat{\tau}_\nu$ is the desired control input w.r.t. the new set V, computed on the estimation of the uncertainties $\hat{\eta}_a$. Hence, the control $\hat{\tau}$ is the actual torque control applied to the system.

Consider the desired torque control law (13), that take care of unknown dynamic parameter η_d with the adaptation law (15). Recalling the actual torque control $\hat{\tau} = \tau + \tilde{\tau}$ and equation (14) we obtain:

$$\begin{aligned} B(\eta_d)\dot{\tilde{u}} &= W(q)(\tau + \tilde{\tau}) - C(\eta_d)u_\tau - G(\eta_d) + \\ &\quad + \gamma^\star(\eta_d) - B(\eta_d)\dot{u}_\tau \\ &= -K_b\tilde{u} - \Delta + Y\tilde{\eta}_d - \gamma\tilde{\eta}_d - C(\eta_d)\tilde{u} + W\tilde{\tau} \end{aligned} \tag{18}$$

Adding the uncertainties on the actuators η_a, the derivative of the Lyapunov function (17) is no longer defined. Hence, it is necessary to complete $\mathbf{V}_3(q, \tilde{u}, \tilde{\eta}_d)$ with an addtional:

$$\begin{cases} \mathbf{V}_{\eta_a}(\eta_a, \tilde{\eta}_a) = \dfrac{1}{2}\bar{\mathbf{1}}^T\tilde{\Theta}^T_{\eta_a}\Theta^{-T}_{\eta_a}\Gamma_a\tilde{\Theta}_{\eta_a}\bar{\mathbf{1}} > 0 \\ \dot{\mathbf{V}}_{\eta_a}(\eta_a, \tilde{\eta}_a) = \bar{\mathbf{1}}^T\tilde{\Theta}^T_{\eta_a}\Theta^{-T}_{\eta_a}\Gamma_a\dot{\tilde{\Theta}}_{\eta_a}\bar{\mathbf{1}} \end{cases}$$

where $\Gamma_a > 0$, symmetric and depends on the sign of the actuator uncertainties η_a. Analogously to (6), $\tilde{\tau} = T\tilde{\Theta}_{\eta_a}\Theta^{-1}_{\eta_a}\bar{\mathbf{1}}$, with $T = \mathrm{diag}\tau_i$. The actuators parameters adaptation law can be chosen as:

$$\dot{\tilde{\Theta}}_{\eta_a}\bar{\mathbf{1}} = \dot{\tilde{\eta}}_a = \dot{\hat{\eta}}_a = -\Gamma_a^{-1}T^TW^T\tilde{u} \tag{19}$$

and constructing the new Lyapunov function $\mathbf{V}_4(q, \tilde{u}, \tilde{\eta}_d, \eta_a, \tilde{\eta}_a) = \mathbf{V}_3(q, \tilde{u}, \tilde{\eta}_d) + \mathbf{V}_{\eta_a}(\eta_a, \tilde{\eta}_a)$, yields:

$$\begin{aligned} &\dot{\mathbf{V}}_4(q, \tilde{u}, \tilde{\eta}_d, \eta_a, \tilde{\eta}_a) = \dot{\mathbf{V}}_1(q) - \tilde{u}^TK_b\tilde{u} + \\ &-\tilde{u}^TWT\tilde{\Theta}_{\eta_a}\Theta^{-1}_{\eta_a}\bar{\mathbf{1}} - \bar{\mathbf{1}}^T\tilde{\Theta}^T_{\eta_a}\Theta^{-T}_{\eta_a}T^TW^T\tilde{u}) \\ &= \dot{\mathbf{V}}_1(q) - \tilde{u}^TK_b\tilde{u} = \dot{\mathbf{V}}_2(q, \tilde{u}) \end{aligned}$$

that is negative semidefinite, with equilibrium point $(q, \tilde{u}, \tilde{\eta}_d, \tilde{\eta}_a) = (0, 0, \tilde{\eta}_d, \tilde{\eta}_a)$.

The most powerful feature of the proposed approach is the design independence between the problems involved in the stabilization task, whose feasibility is achieved using backstepping techniques and computed torque frameworks. The controller design steps could be depicted briefly in what follows:

1. Design a desired control law for the kinematic nonholonomic system $u(q)$;
2. Starting from $u(q)$, design the desired torque τ for the dynamic system (see (10));
3. If there are uncertainties on the dynamic parameters η_d of the system, modify the torque control law τ (and τ_ν) with estimated values and use the dynamic parameters adaptation law (15);
4. If there are uncertainties on the actuator parameters η_a too, add the actuator parameters adaptation law (19).

5. TRACKING CONTROL WITH BOUNDED TORQUES.

The general, modular, framework presented in the previous paragraphs is now extended to the case of constraints in the actuators torques for the case of unicycle motion. An *approaching controller* is used to minimize the distance e between the controlled vehicle and the reference vehicle under a parameter dependent threshold $D_{\min}$. As the robot reaches the desired distance $D_{\min}$, another parameter dependent distance $D_{\max} > D_{\min}$ is defined and a *backstepping controller* is activated. The latter controller stabilizes the robot onto the desired trajectory and guarantees that the control torques are limited if the distance $e < D_{\max}$ (this condition is guaranteed by the backstepping controller as well).

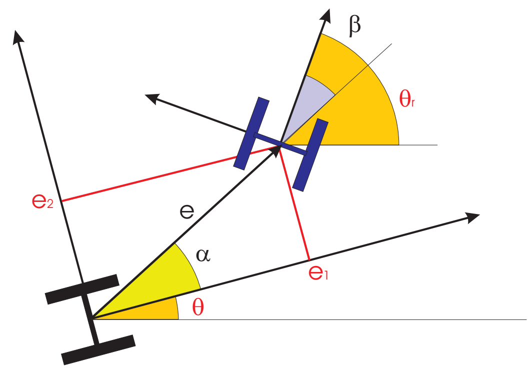

Fig. 1. Trajectory tracking problem geometry.

5.1 Backstepping Controller

Let us consider the kinematic and dynamic model of a unicycle and express the target coordinates in the mobile vehicle reference system. Hence, the error dynamics $e = [e_1, e_2, e_3]^T$ are given by the equations:

$$\begin{cases} \dot{e}_1 = v_r\cos(e_3) - v + e_2\omega \\ \dot{e}_2 = v_r\sin(e_3) - e_1\omega \\ \dot{e}_3 = \omega_r - \omega \end{cases} \Rightarrow \dot{e} = f(e) + g(e)u \tag{20}$$

where v and ω are the forward and steering velocities. Dynamics are added by considering the equations $\dot{v}(e) = \tau_v/m$ and $\dot{\omega}(e) = \tau_\omega/I_z$, where m and I_z are the mass and the momentum of inertia respectively (see (Aguiar *et al.*, 2000)).

Using the control Lyapunov function

$$\mathbf{V}_1 = \frac{1}{2}\left(e_1^2 + e_2^2\right) + (1 - \cos(e_3)) \tag{21}$$

we are able to synthesize two kinematic control laws for the forward and steering velocities, $\bar{v}$ and $\bar{\omega}$ respectively

$$\begin{aligned} \bar{v}(e) &= \frac{2}{\pi}V_{\max}\arctan(e_1) - v_r\cos(e_3) \\ \bar{\omega}(e) &= \omega_r + \frac{1}{K_{e_3}}e_2v_r + \frac{1}{K_{s_3}}\sin(e_3) \end{aligned} \tag{22}$$

that make $\dot{\mathbf{V}}_1$ negative definite. It is easy to notice that the forward velocity $\bar{v}$ is surely bounded by $V_{\max} + v_r$, while $\bar{\omega}$ maximum value depends on the positive constants K_{e_3}, K_{s_3}, but also on the distance, through the error variable e_2.

Adding dynamics, a new control Lyapunov function is obtained as:

$$\mathbf{V}_1^d(e,v,\omega) = \mathbf{V}_1 + \frac{1}{2}(v-\bar{v})^2 + \frac{1}{2}(\omega-\bar{\omega})^2$$

that yields

$$\begin{aligned}\tau_v &= m(-K_{bv}(v-\bar{v}) + \frac{\partial\bar{v}}{\partial e}\dot{e} + \frac{\partial \mathbf{V}_1}{\partial e}g_v(e)) \\ \tau_\omega &= I_z(-K_{b\omega}(\omega-\bar{\omega}) + \frac{\partial\bar{\omega}}{\partial e}\dot{e} + \frac{\partial \mathbf{V}_1}{\partial e}g_\omega(e))\end{aligned} \quad (23)$$

where K_{bv} and $K_{b\omega}$ are positive constants, $\bar{v}$ and $\bar{\omega}$ are the reference velocities provided by the kinematic controller, and $f(e)$ and $g(e)$ are again the vector fields of the kinematic model (20).

We are now interested in finding a maximum for the two torques (23):

$$\begin{aligned}&|\tau_{v_{\max}}| = m(2K_{bv}|v_{\max}| + |v_{r_{\max}}|(|\omega_{\max}| + \\ &+|\omega_{r_{\max}}|) + |v_{r_{\max}}| + |v_{\max}| + |d||\omega_{\max}| + |d|) = \\ &= T_{v_1} + d\cdot T_{v_2} \\ \\ &|\tau_{\omega_{\max}}| = I_z(2K_{b\omega}|\omega_{\max}| + K_{e_3}^{-1}(v_{r_{\max}}^2 + \\ &+|v_{r_{\max}}||\omega_{\max}|\cdot d) + K_{s_3}^{-1}(|\omega_{\max}| + |\omega_{r_{\max}}|) + \\ &+K_{e_3}) = T_{\omega_1} + d\cdot T_{\omega_2}\end{aligned} \quad (24)$$

As shown in equations (24), the maximum value of the torques are given by a linear relation $|\tau_{\max}| = T_1 + d\cdot T_2$, where T_1 and T_2 are functions of the vehicle maximum velocities and inertial parameters. The lower limit of the torque value is T_1, hence the problem to solve is to find the maximum value of d in order to constraint the torque into the range $[0, T_1 + \Delta_{\max}]$, with $\Delta_{\max} > 0$. Since the torque controls critically depend on the distance between the controlled and the desired reference vehicle, before the resulting backstepping torque controls can be applied to the system, an additional controller, approaching the desired reference vehicle, is adopted.

5.2 Approaching Controller

This controller is meant to drive the vehicle inside the range where the backstepping controller works with constrained torques. Consider a new state space q for the vehicle (see again figure 1), where $q = [e, \alpha, \beta]^T$ with e the distance vector between the vehicle and the target and with α and β angles between vector e and the relative direction of each vehicle. The kinematic model of the variables q is then:

$$\begin{cases}\dot{e} = -v\cos\alpha + v_r\cos\beta \\ \dot{\alpha} = -\omega + v\dfrac{\sin\alpha}{e} + v_r\dfrac{\sin\beta}{e} \\ \dot{\beta} = \omega_r - v\dfrac{\sin\alpha}{e} - v_r\dfrac{\sin\beta}{e}\end{cases} \quad (25)$$

Consider the Control Lyapunov Function

$$\mathbf{V}_2(q) = \frac{1}{2}\alpha^2 + \frac{1}{2}\ln(1+e^2),$$

defined on $(\alpha, e, \beta) = [0, 2\pi) \times [D_{\min}, +\infty] \times [0, 2\pi)$. $\mathbf{V}_2$ does not depend explicitly on β as, during the approaching phase, the relative orientation between the target and the vehicle is not relevant. Its time derivative is given by

$$\dot{\mathbf{V}}_2(q) = \alpha\dot{\alpha} + \frac{e}{1+e^2}\dot{e}$$

and by substituting the controls $\bar{v}$ and $\bar{\omega}$:

$$\begin{cases}\bar{v} = \mathrm{sat}(e)\cos\alpha \\ \bar{\omega} = K_\alpha\alpha + v\dfrac{\sin\alpha}{e} + v_r\dfrac{\sin\beta}{e}\end{cases} \quad (26)$$

where $\mathrm{sat}(e)$ is a saturation function (see (Beard and Ren, 2004)), we obtain

$$\begin{aligned}\dot{\mathbf{V}}_2 &= \frac{e}{1+e^2}\left(-\frac{1+e^2}{e}K_\alpha\alpha^2 - \mathrm{sat}(e)\cos^2\alpha + v_r\cos\beta\right) \\ &= \frac{e}{1+e^2}\dot{\tilde{\mathbf{V}}}_2\end{aligned}$$

As $\mathrm{sgn}(\dot{\mathbf{V}}_2) = \mathrm{sgn}(\dot{\tilde{\mathbf{V}}}_2)$, let us maximize $\dot{\tilde{\mathbf{V}}}_2$ (recall that $e \geq D_{\min}$)

$$\dot{\tilde{\mathbf{V}}}_2 \leq -2K_\alpha\alpha^2 - \mathrm{sat}(e)\cos^2\alpha + v_{r_{\max}}.$$

$\mathrm{sat}(e)$ is an increasing function of the distance, therefore, its minimum point is $D_{\min}$. It is always possible to choose $\mathrm{sat}(e)$ such that $v_{D_{\min}} = \mathrm{sat}(D_{\min}) > v_{r_{\max}}$, necessary condition for $\dot{\tilde{\mathbf{V}}}_2$ to be negative definite on $\alpha = 0$. Hence

$$\dot{\tilde{\mathbf{V}}}_2 \leq -2K_\alpha\alpha^2 - v_{D_{\min}}\cos^2\alpha + v_{r_{\max}}.$$

$-v_{D_{\min}}\cos^2\alpha + v_{r_{\max}}$ can be positive for $\alpha \in (\underline{\alpha}, \overline{\alpha})$ with $\underline{\alpha} = \mathrm{acos}\left(-\sqrt{\frac{v_{r_{\max}}}{v_{D_{\min}}}}\right)$ and $\overline{\alpha} = \mathrm{acos}\left(\sqrt{\frac{v_{r_{\max}}}{v_{D_{\min}}}}\right)$. Note that $\underline{\alpha}$ and $\overline{\alpha}$ are well defined since $v_{D_{\min}} > v_{r_{\max}}$. Hence a sufficient condition for $\dot{\tilde{\mathbf{V}}}_2$ to be negative definite is given by

$$-2K_\alpha\underline{\alpha}^2 + v_{r_{\max}} < 0 \Rightarrow K_\alpha > \frac{v_{r_{\max}}}{2\underline{\alpha}^2}.$$

With this parameter choice the system (25) with the control laws (26) is globally uniformly ultimately bounded on $(\alpha, e) = [0, 2\pi) \times [D_{\min}, +\infty]$.

As in (23), backstepping techniques lead to the control laws for the vehicle's dynamic model, that can be maximized as in (24):

$$\begin{aligned}|\tau_{v_{\max}}| &= m(2K_{bv}|v_{\max}| + \frac{R_{\max}}{\pi}(|v_{r_{\max}}| + |v_{\max}|) + \\ &\quad +\pi(1 + K_\alpha + \frac{1}{2\pi})) \\ |\tau_{\omega_{\max}}| &= I_z(2K_{b\omega}|\omega_{\max}| + (|v_{r_{\max}}| + |v_{\max}|)^2 + \\ &\quad +K_\alpha(|v_{\max}| + K_\alpha) + |v_{r_{\max}}|(|\omega_{r_{\max}}| + \\ &\quad +|v_{\max}| + |v_{\max}|) + \pi)\end{aligned} \quad (27)$$

The control laws (27) allow to bound the control torques while the vehicle approaches the region where the controller (23) is able to track the target and respect the torques constraints.

5.3 Switching Control Law

The adopted switching control law is very simple: just a single switch from the approaching to the backstepping controller is allowed once the distance between the vehicle and the target is less than $D_{\min}$. The convergence of the proposed method is proven considering that the approaching controller ensures that the region with $e < D_{\min}$ is reached in a finite time, where the asymptotically stable backstepping controller is activated.

We are now interested in fixing the value of the switching distance $D_{\min}$: let us consider an isosurface of the Lyapunov function $\mathbf{V}_1$. The projections on the (e_1, e_2) plane for different e_3 angles are concentric circles. From Lyapunov theory, a system trajectory originated inside an isosurface with n.d. time derivative is bounded in the same isosurface. This means that fixing $D_{\min}$ as the radius of the smallest circle (computed at $e_3 = \pi$), in the worst case, when the backstepping controller is activated, a trajectory starting at a distance $D_{\min}$ from the origin of the (e_1, e_2) plane will never exceed the $D_{\max}$ distance, radius of the largest circle (computed at e_3=0), therefore the torque constraints will be respected.

6. SIMULATION RESULTS

6.1 Regulation with kinematic uncertainties and limited velocity

As an application example of the method explained previously, a unicycle like vehicle is controlled with limited velocity. The kinematic control law reported in (Murrieri *et al.*, 2004) has been modified as reported in (Caiti *et al.*, 2005).

In figure 2 is depicted a parking trajectory when the robot is placed in $q = [270, 3.6, 3.95]^T$ and the vehicle parameters are set to $R = 100$ and $L = 500$. The estimated parameters value is $\hat{R} = 190$ and $\hat{L} = 150$. The velocity limits are $(v_{\max}, \omega_{\max}) = (50\ mm/sec,\ 1/2\ rad/sec)$, while the scaling factor $\rho_0 = 660$. The controller parameter λ in (Murrieri *et al.*, 2004) is 0.04. The parking problem is solved using the adaptive controller with limited velocity.

In figure 3, the computed controls are reported (the linear velocity v, left, and the angular velocity ω, right). Each figure depicts also the velocity limits.

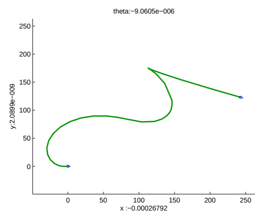

Fig. 2. Vehicle manoeuvre during a docking operation with unknown parameters and limited velocities.

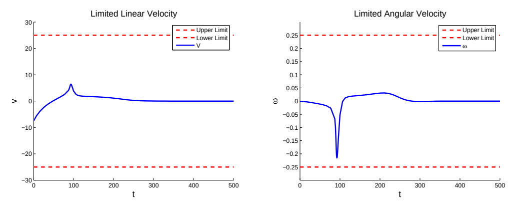

Fig. 3. Computed controls: linear velocity v (left), angular velocity ω (right). Each figure reports the velocity limits too.

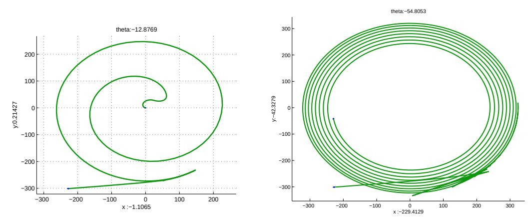

Fig. 4. Vehicle manoeuvre during a docking operation with both unknown dynamic and actuator's parameters. Both the adaptive (left) and native (right) controller behavior is reported.

6.2 Regulation with dynamic and kinematic uncertainty

Let $\tau = [\tau_v, \tau_\omega]^T = [\tau_1, \tau_2]^T$ be the available force and torque controls, i.e. the forward force and the steering torque of the vehicle respectively, and $\tau_\nu = [\tau_r, \tau_l]^T = [\tau_{\nu_1}, \tau_{\nu_2}]^T$ be the torques of the vehicle's wheels, on the right and left side of the robot respectively:

$$\begin{cases} \tau_v = (\tau_r + \tau_l)\dfrac{1}{R} \\ \tau_\omega = (\tau_r - \tau_l)\dfrac{L}{2R} \end{cases} \Leftrightarrow \begin{cases} \tau_r = \tau_v \dfrac{R}{2} + \tau_\omega \dfrac{R}{L} \\ \tau_l = \tau_v \dfrac{R}{2} - \tau_\omega \dfrac{R}{L} \end{cases}.$$

In figure 4 is depicted a parking trajectory when the robot is placed in $q = [380, 0.92, 3.97]^T$, with the mass $m = 10$ and the inertia momentum $I = 1$ and with the actuator's parameters set to $R = 100$ and $L = 500$. The dynamic estimated parameters $\hat{\eta}_d = [\hat{m}, \hat{I}]^T = [76, 4.5]^T$ and the actuator's estimated parameters $\hat{\eta}_a = [1/\hat{R}, \hat{L}/\hat{R}]^T = [1/174, 664/174]^T$. The controller parameter $\lambda = 1/2$. The simulation has been carried out for $20 sec$. On the left, the parking problem is solved using the adaptive controller while, on the right side of the figure, the parking problem is carried out with the native controller $u(q)$, without any

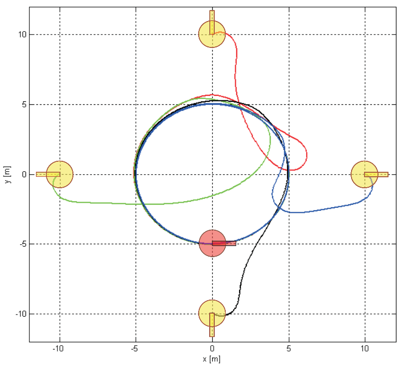

Fig. 5. Trajectories of mobile robot at different starting position.

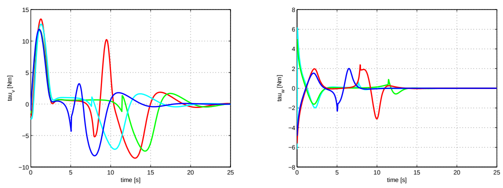

Fig. 6. Computed torque controls: linear (left) and angular (right) controls.

parameter adaptation (see (Caiti *et al.*, 2005) for more detailed simulation results).

6.3 Tracking control with bounded torques

Simulation results for the tracking control of a target vehicle with bounded torques are presented. In what follows, the reference vehicle describes a circle while the controlled vehicle starts from different positions. Vehicle parameters can be found in (Caiti *et al.*, 2005).

In figure 5 the trajectories of the controlled robot starting from $q = [D\cos\delta, D\sin\delta, \delta]^T$, with $D = 10m$ and $\delta \in \{0, \pi/2, \pi, -\pi/2\}$ are depicted. The target starts from $q_r = [0, -5, 0]^T$ and describes a circle of radius $r = 5m$. The corresponding computed torque controls are reported in figure 6.

7. CONCLUSIONS

Nonlinear adaptive control laws for generic kinematic nonholonomic systems in the presence of actuator limits and uncertainties have been derived. An extension to uncertain dynamic systems using backstepping techniques and control Lyapunov functions has been used. It has been shown that it is possible to obtain a control Lyapunov function in a modular way, starting from a stabilizing law for the kinematic, perfectly known model. Our effort has been devoted to bound the control inputs of the kinematic and dynamic system, in order to avoid actuator saturations.

Simulation results have been reported, showing a vehicle manoeuvre during a parking operation with unknown parameters and limited velocities and vehicle manoeuvres during a tracking operation with limited torques. Our future efforts will focus on merging the previous results and the obtaining smooth control laws for the set-point and tracking control problems.

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