Controllability for Pairs of Vehicles Maintaining Constant Distance

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Abstract—This paper studies the controllability of pairs of identical nonholonomic vehicles maintaining a constant distance. The study provides controllability results for the five most common types of robot vehicles: Dubins, Reeds-Shepp, differential drive, car-like and convexified Reeds-Shepp. The challenge of achieving controllability of such systems is that their admissible control domains depend on configuration variables. A theorem of controllability specific for such systems has been obtained based on known controllability theorems. As a result, we show that pairs of the latter three types are completely controllable, i.e. can be steered between any two arbitrary configurations. The same does not hold for pairs of Dubins or Reeds-Shepp vehicles, and a description of the reachable sets in these cases is provided. Finally, as direct extension of controllability results of pairs of identical vehicles, the controllability results for two kinds of formation of $n$ identical vehicles are presented.

I. INTRODUCTION

This paper provides the results of controllability for pairs of identical vehicles maintaining a constant distance. The controllability of a system answers the question about the existence of an admissible trajectory between any given two configurations, which is an important condition for a feasible design of motion planning ([1]) and for the existence of an optimal trajectory (see e.g. [2]). Moreover, the study of pairs of vehicles maintaining a constant distance helps the design of navigation strategies for a group of robots moving in formation (see e.g. [3], [4] and [5]).

In this paper we adopt the notation used in [2], [6]. A system is controllable if, for every pair of points $p$ and $q$ in the configuration space, there exists a control that steers the system from $p$ to $q$. It is small-time locally controllable (STLC) from a point $p$ if the set of points reachable before a given time $T$ contains a neighborhood of $p$ for any $T$. A control system will be said to be small-time controllable if it is small-time controllable from any point of the configuration space. The small-time controllability can be used to answer the problem about existence of collision-free admissible paths (see e.g. [1]). The challenging aspect in the controllability of the considered systems is that admissible controls depend on the configuration variables. Therefore, based on existing controllability theorems and on the accessibility rank condition of weakly reversible systems, we provide controllability theorems specific for such systems. Furthermore, conditions to verify the controllability of such systems are also provided.

This paper provides the results of controllability for the five most common types of robot vehicles which are widely discussed in the literature: Dubins [7], Reeds-Shepp (RS) [8], differential drive (DDV) [9], [10], car-like (Car) [11], [12] and convexified Reeds-Shepp (CRS) [2]. As a result, we show that while pairs of the latter three types of vehicle can be steered between any two arbitrary configurations, the same does not hold for pairs of vehicles of the first two types. For these two cases, a description of the reachable sets is provided. To the authors’ best knowledge, in the current literature no result on the controllability of pairs of vehicles that maintain a given distance is reported.

II. CONTROLLABILITY THEOREMS

We first introduce the controllability theorems and lemmas that we will use in the following sections to prove controllability for the considered systems.

A. Controllability Definitions and Theorems

The systems we will study are affine control systems that can be written as

$$\Sigma_{\text{aff}} \colon \left\{ \begin{array}{l}
\dot{x} = f(x, u) = g_0(x) + \sum_{i=1}^{m} g_i u_i; \\
x \in \mathcal{X} \subseteq \mathbb{R}^n, u \in \mathcal{U}(x) \subseteq \mathbb{R}^m.
\end{array} \right. \quad (1)$$

Let $\mathcal{A} := \{f_u = f(\cdot, u), u \in \mathcal{U}\}$ be the set of system’s vector fields.

Definition 1: The Lie algebra $\mathcal{A}_{LA}$ of vector fields $\mathcal{A}$ is called the accessibility Lie algebra associated to the system. The accessibility rank condition (ARC) holds at $x_0 \in \mathcal{X}$ if $\mathcal{A}_{LA}(x_0) = \mathbb{R}^n$.

Accessibility rank condition in [13] is also called controllability rank condition in [14], and Lie algebra rank condition...
that exists an admissible control \( R \) a sufficient condition for set it is possible to reach points in which the ARC holds: (based on a trivial extension of Theorem 2) whenever from this

Recall that a system is symmetric if every trajectory run backwards in time is also a trajectory.

**Theorem 1:** For a symmetric system, if the accessibility rank condition holds at every point \( x_0 \in \mathcal{X} \), the system is STLC from every \( x_0 \). In particular, if \( \mathcal{X} \) is connected, then it is controllable. Moreover, if it is symmetric, then it is also STLC.

Theorem 3: Given a weakly reversible affine control system, such as 1, with \( \mathcal{X} \) connected, and given \( S^1 \subset \mathcal{X} \) such that

1. \( \forall x_0 \in S^1 \), \( U(x_0) \subseteq \mathbb{R}^m \) almost proper, \( A_{LA}(x_0) = \mathbb{R}^n \);
2. \( \forall x_0 \in S^2 := \mathcal{X} \setminus S^1, U(x_0) \subseteq \mathbb{R}^l, l < m, A_{LA}(x_0) \neq \mathbb{R}^n \), but \( \mathcal{R}(x_0) \cap S^1 \neq \emptyset \),
then the system is completely controllable. Moreover, if it is symmetric, then it is also STLC.

**Remark 1:** Whenever \( S^2 \) does not have interior points and it is such that its boundary function \( \Phi(x) \) is differentiable, a sufficient condition for \( \mathcal{R}(x_0) \cap S^1 \neq \emptyset \) is that there exists an admissible control \( \omega \in U(x_0) \), \( x_0 \in S^2 \) such that \( \{ f(x_0, \omega), \frac{\partial f}{\partial \omega} \} \neq 0 \). This condition will be used to prove the controllability of Car and RS vehicles. And if \( \{ f(x_0, \omega), \frac{\partial f}{\partial \omega} \} = 0 \), \( S^2 \) is invariant under all admissible control \( \omega \), hence the system is not controllable, see fig.2.

### III. Kinematic Models

In this section the kinematic model for two identical vehicles (Dubins, Reeds-Shpeh, differential drive, car-like and convexified Reeds-Shpeh) traveling at constant distance is obtained starting from the kinematic model of a single vehicle. It is worthwhile noticing that the models will differ in the control set and not in the kinematics.

**A. Kinematic Models for Single Vehicles**

The kinematic model of the considered vehicles can be described as

\[
\begin{pmatrix}
  x_i \\
  y_i \\
  \dot{\theta}_i
\end{pmatrix} = \begin{pmatrix}
  \cos \theta_i \\
  \sin \theta_i \\
  0
\end{pmatrix} u_i + \begin{pmatrix}
  0 \\
  0 \\
  1
\end{pmatrix} v_i
\]

(2)

where \( \xi_i = (x_i, y_i, \theta_i) \in \mathbb{R}^2 \times S^1 \) denotes a configuration of vehicle \( i \), i.e. \( (x_i, y_i) \) is the position and \( \theta_i \) is the forward direction angle with respect to the positive x-axis.

The controls \( u_i \) and \( v_i \) describe the linear and angular velocities of vehicle \( i \), respectively. We write \( (u_i, v_i) \in U \), where \( U \) is the admissible control domain, Fig. 1 shows the different admissible control domains for the above five types of vehicles. Without loss of generality, we consider normalized maximal and minimal velocities and assume that the minimum turning radius \( R_{\text{min}} = 1 \) for Dubins, RS and car-like robots, although in order to emphasize its influence \( R_{\text{min}} \) often remains.

For DDV the wheel angular velocities are bounded, hence the admissible control domain is a diamond (rhombus), i.e. \( U_{\text{DDV}} = \{(u_i, v_i)|0 \leq |u_i| \leq 1|0 \leq |v_i| \leq 1\} \). For car-like vehicles, \( U_{\text{car}} = \{(u_i, v_i)|0 \leq |v_i| \leq 1\} \). A Dubins vehicle is a car-like vehicle which is only able to move forward with constant velocity, i.e. \( U_{\text{Dubins}} = 1 \times [−1, 1] \). RS vehicles, can move both forward and backward at constant velocity 1, i.e. \( U_{\text{RS}} = \{-1, 1\} \times [−1, 1] \). For CRS robots, \( U_{\text{CRS}} = [-1, 1] \times [-1, 1] \) is obtained by convexifying \( U_{\text{RS}} \) and CRS is the kinematic model of a tricycle.

**B. Kinematic Models for Pairs of Vehicles**

Consider a pair of vehicles \( (\xi_1, \xi_2) \) traveling while maintaining a constant distance \( D \). Let \( \phi \) denote the angle of vector \( (x_2 - x_1, y_2 - y_1) \) with respect to the x-axis, see fig.3. Thus we can write:

\[
x_2 - x_1 = D \cos \phi; \quad y_2 - y_1 = D \sin \phi.
\]

(3)

We choose \( \xi_{1-2} = (x_1, y_1, \theta_1, \phi, \theta_2) \) as the configuration vector of the system consisting of two identical vehicles maintaining a constant distance. The nonholonomic constraint for each vehicle is:

\[
\dot{x}_1 \sin \theta_1 - \dot{y}_1 \cos \theta_1 = 0.
\]

From (3), we have that \( \dot{x}_2 = -D \dot{\phi} \sin \phi + \dot{x}_1 \) and \( \dot{y}_2 = D \dot{\phi} \cos \phi + \dot{y}_1 \).
Hence, from (2) and (6), we obtain that
\[
f_5 = \begin{pmatrix} D \cos \theta_1 \cos \gamma_2 \\ D \sin \theta_1 \cos \gamma_2 \\ \sin(\gamma_1 - \gamma_2) \\ \sin(\gamma_1 - \gamma_2) \\
\end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\
\end{pmatrix} v_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\
\end{pmatrix} v_2. 
\]

Let \( q = (x_1, y_1, \theta_1, \gamma_1, \gamma_2) \) be the new configuration of the system with \( \gamma_1 = \phi - \theta_1, \gamma_2 = \phi - \theta_2 \). The kinematic model of a pair of identical vehicles maintaining distance \( D \) can be written as:

\[
\dot{q} = f_1 u + f_2 u_1 + f_3 u_2,
\]

where the system vector fields are:

\[
f_1 = \begin{pmatrix} D \cos \theta_1 \cos \gamma_2 \\ D \sin \theta_1 \cos \gamma_2 \\ \gamma_1 - \gamma_2 \\ \gamma_1 - \gamma_2 \\
\end{pmatrix} ; \quad f_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\
\end{pmatrix} ; \quad f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\
\end{pmatrix}.
\]

The relationship between \( u \) in (6) and \( u_1 \) and \( u_2 \) must be found. To maintain distance \( D \), the velocity of both vehicles along distance direction should be the same, i.e.

\[
u_1 \cos \gamma_1 = u_2 \cos \gamma_2.
\]

Hence, from (2) and (6), we obtain that

\[
u_1 = u D \cos \gamma_2; u_2 = u D \cos \gamma_1.
\]

The systems of pairs of identical vehicles maintaining a constant distance will be denoted by \( \Sigma_{DDV}, \Sigma_{Car}, \Sigma_{Dubins}, \Sigma_{RS} \) and \( \Sigma_{CRS} \) for differential drive, car-like, Dubins, RS and CRS vehicles, respectively.

\[\text{Fig. 3. The kinematic model of pairs of identical vehicles.}\]

Finally, the constraints for a pair of vehicles maintaining distance \( D \) can be written as:

\[
\begin{align*}
\dot{x}_1 \sin \theta_1 - \dot{y}_1 \cos \theta_1 &= 0 \\
-D\dot{\phi} \sin \phi + \dot{x}_1 \sin \theta_2 - (D\dot{\phi} \cos \phi - \dot{y}_1) \cos \theta_2 &= 0.
\end{align*}
\]

Hence, a system of 5 unknowns \((\dot{x}_1, \dot{y}_1, \dot{\phi}, \dot{\theta}_1, \dot{\theta}_2)\) and 2 linear equations (4) have been obtained. By computing the null space of the constraint matrix we obtain that there exists \( u \) such that the kinematic model of a pair of identical vehicles maintaining distance \( D \) is:

\[
\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\phi} \\ \dot{\theta}_1 \\ \dot{\theta}_2 
\end{pmatrix} = \begin{pmatrix} D \cos \theta_1 \cos \phi - \partial_2 \\ D \sin \theta_1 \cos \phi - \partial_2 \\ \sin(\theta_1 - \theta_1) \\ 0 \\ 0 
\end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 
\end{pmatrix} v_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 
\end{pmatrix} v_2. 
\]

\[\text{IV. CONTROLLABILITY FOR DDV, CAR AND CRS SYSTEMS}\]

\[\text{A. Controllability for DDV}\]

**Theorem 4:** \( \Sigma_{DDV} \) is STLC and controllable on the configuration space \( \mathcal{M}_{DDV} = \mathbb{R}^2 \times S^1 \times S^1 \times S^1 \).

**Proof:** For DDV, the control is \( |u| \leq 1 \), and no constraint limits the configuration variables \( \theta_1, \gamma_1 \) and \( \gamma_2 \). Hence the configuration space is \( \mathcal{M}_{DDV} = \mathbb{R}^2 \times S^1 \times S^1 \times S^1 \).

Moreover, from (8) and \( |u_1| \leq 1 \) it follows

\[
\begin{align*}
|u_1| &\leq \min \left\{ \frac{|\cos \gamma_2|}{|\cos \gamma_1|}, 1 \right\}; \\
|u_2| &\leq \min \left\{ \frac{|\cos \gamma_1|}{|\cos \gamma_2|}, 1 \right\}.
\end{align*}
\]

(10)

From \( |v_i| \leq 1 \), we obtain that \( |u| \leq 1 \). Note that \( |u| \leq 1 \), its admissible control set is \( \mathcal{U}_{DDV} = \{(v_1, v_2, u) ||v_1| \leq 1; |v_2| \leq 1 \} \), shown in fig. 4 (a); otherwise, \( \mathcal{U}_{DDV} = \{(v_1, v_2, u) ||v_1| \leq 1; |v_2| \leq \min \left\{ \frac{1-|v_1|}{|\cos \gamma_2|}, \frac{1-|v_2|}{|\cos \gamma_1|} \right\} \} \), shown in fig. 4 for two kinds of configuration with (b): \( \gamma_1 = \frac{\pi}{6}, \gamma_2 = \frac{\pi}{2} \); (c): \( \gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{2} \). Thus for any configuration, the system is controllable.

\[\text{B. Controllability for CRS}\]

**Theorem 5:** \( \Sigma_{CRS} \) is STLC and controllable on the configuration space \( \mathcal{M}_{CRS} = \mathbb{R}^2 \times S^1 \times S^1 \).

**Proof:** \( \mathcal{M}_{CRS} = \mathbb{R}^2 \times S^1 \times S^1 \) follows directly from \( |u| \leq 1 \) with the same reasoning used in the previous theorem.

Similarly to the proof of Theorem 4, the admissible control set is \( \mathcal{U}_{CRS} = \{(v_1, v_2, u) ||v_1| \leq 1; |v_2| \leq 1 \} \), shown in fig. 5 (a); otherwise, \( \mathcal{U}_{CRS} = \{(v_1, v_2, u) ||v_1| \leq 1; |v_2| \leq \min \left\{ \frac{1-|v_1|}{|\cos \gamma_2|}, \frac{1-|v_2|}{|\cos \gamma_1|} \right\} \} \), shown in fig. 5 (b) at a specified configuration. The admissible control set \( \mathcal{U}_{CRS} \) is proper for all configurations, \( \Sigma_{CRS} \) is symmetric and \( \mathcal{M}_{CRS} \) is connected. Hence the thesis.

\[\text{C. Controllability for Car}\]

**Theorem 6:** \( \Sigma_{car} \) is STLC and controllable on the configuration space \( \mathcal{M}_{car} = \mathbb{R}^2 \times S^1 \times S^1 \times S^1 \).
Thus \( \Phi(U | U) \)

\[ \text{Fig. 7. The admissible } U_{CRS} \text{ at configurations with (a): } \gamma_1 = \gamma_2 = \frac{\pi}{2}; \]

(b): \( \gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{4} \); and (c): \( \gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{4} \).

\[ \text{Fig. 6. The admissible } U_{car} \text{ at configurations with (a): } \gamma_1 = \gamma_2 = \frac{\pi}{2}; \]

(b): \( \gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{4} \); and (c): \( \gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{4} \).

\[ \text{Fig. 5. The admissible } U_{CRS} \text{ at configurations with (a): } \gamma_1 = \gamma_2 = \frac{\pi}{2}; \]

(b): \( \gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{4} \); and (c): \( \gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{4} \).

\[ \text{Proof: } M_{car} = \mathbb{R}^2 \times S^1 \times S^1 \times S^1 \]

follows directly from \( |u_i| \leq 1 \). From (10) and \( |v_i| \leq |u_i| \leq 1 \), we can get that if \( \gamma_1 = \gamma_2 = \frac{\pi}{2} \),

\[ U_{car} = \{(v_1, v_2, u)| |v_i| \leq 1; |u| \leq 1\}; \]

otherwise \( U_{car} = \{(v_1, v_2, u)| |v_i| \leq \min \{\frac{\cos \gamma_i}{|\cos \gamma_1|}, 1\}; |v_2| \leq \min \{\frac{\cos \gamma_i}{|\cos \gamma_2|}, 1\}\}; \)

\[ |u| \leq \min \{\frac{1}{|\cos \gamma_2|}, \frac{1}{|\cos \gamma_1|}\}. \]

As shown in fig.6, \( U_{car} \)

is almost proper at all configurations except at \( \gamma_i = \frac{\pi}{2} \) and \( \gamma_j \neq \frac{\pi}{2}, i, j = 1, 2 \). For such configurations \( U \) is shown in fig. 6(c) and \( \text{aff}(U_{car}) = \mathbb{R}^2 \). Considering \( \gamma_1 = \frac{\pi}{2}, \gamma_2 \neq \frac{\pi}{2} \) (see fig.7), we have \( \Phi(x) = \gamma_1 - \frac{\pi}{2} \). If we choose \( u_1 = 1, v_1 = -1 \), then \( f(x_0, \omega) = (*, *, *, +, D + 1, *) \) and \( \frac{\partial f}{\partial x} = (0, 0, 0, 1, 0, 1) \).

Thus \( f(x_0, \omega) > 0 \). Thus the thesis follows from Theorem 3.

V. CONTROLLABILITY FOR RS

For RS vehicles, \( u_i = \pm 1 \). Hence, from (8), we have:

\[ \cos \gamma_1 = \pm \cos \gamma_2. \]

(11)

Four possible angular relationships between two vehicles can thus be obtained (see fig. 8):

\[ \gamma_1 = \gamma_2 \]

\[ \gamma_1 = -\gamma_2 - \pi \]

\[ \gamma_1 = -\gamma_2 \]

\[ \gamma_1 = \gamma_2 - \pi \]

\[ \text{Fig. 8. Four angular relationships for } \Sigma_{RS} \text{ and configuration representation by 4-dimensional parameters plus angular relationships } \{a_1, a_2, b_1, b_2\}. \]

\[ \gamma_1 = \gamma_2 \]

\[ \gamma_1 = -\gamma_2 \]

\[ \gamma_1 = \gamma_2 - \pi \]

\[ \gamma_1 = -\gamma_2 - \pi \]

\[ \text{Fig. 9. Four angular relationships for pairs of RS vehicles and the combined cases.} \]

\[ a_1 : \gamma_1 = \gamma_2; \quad a_2 : \gamma_1 = \gamma_2 - \pi; \]

\[ b_1 : \gamma_1 = -\gamma_2; \quad b_2 : \gamma_1 = -\gamma_2 - \pi. \]

(12)

For simplicity and clarity of configurations representation for \( \Sigma_{RS} \), we reduce the variables to 4 \((x_1, y_1, \theta_1, \gamma_1)\) and we use a parameter \((a_1, a_2, b_1, b_2)\) to denote the angular relationship (12). In fig.9 four possible angular relationships are represented together with the shared cases: \(|\gamma_1| = |\gamma_2| = 0\) and \(|\gamma_1| = |\gamma_2| = \frac{\pi}{2}\).

We denote with \( \Sigma_{RS}^A \) the system \( \Sigma_{RS} \) when relation \( a_1 \) or \( a_2 \) holds (in this case \( v_1 = v_2 \)). From (5), the kinematic model of \( \Sigma_{RS}^A \) is

\[ \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \cos \theta_1 & 0 & 0 & 0 & 1 \\ \sin \theta_1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \end{pmatrix}, \]

(13)
with $u_1 \in \{-1, 1\}$ and $v_1 \in [-1, 1]$.

Remark 2: Notice that $\gamma_1 = -\theta_1$, hence there always exists a control $(u_1, v_1)$ that steers $\gamma_1$ between any two values keeping $\theta = \gamma_1 + \theta_1$ constant.

We denote with $\Sigma_{RS}^B$ the system $\Sigma_{RS}$ when relation $b_1$ or $b_2$ holds (in this case $v_2 = \frac{4\sin \gamma_1}{D} u_1 - v_1$). From $uD\cos \gamma_2 = u_1$ and (11), we have that the kinematic model of $\Sigma_{RS}^B$ is

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
\dot{\theta}_1 \\
\dot{\gamma}_1 \\
\dot{\gamma}_2
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_1 \\
\sin \theta_1 \\
0 \\
\frac{2 \sin \gamma_1}{D} \\
\frac{2 \sin \gamma_1}{D}
\end{pmatrix} u_1 +
\begin{pmatrix}
0 \\
0 \\
1 \\
-1 \\
1
\end{pmatrix} v_1.
\]

(14)

Let the 4-dimensional system $\tilde{\Sigma}_{RS}^B$ be system $\Sigma_{RS}^B$ projected on the first four coordinates. Hence, the configuration of $\Sigma_{RS}^B$ is $\tilde{\gamma} = (x_1, y_1, \theta_1, \gamma_1)$ and the vector fields are

\[
g_1 = \begin{pmatrix}
\cos \theta_1 \\
\sin \theta_1 \\
0 \\
\frac{2 \sin \gamma_1}{D}
\end{pmatrix};
g_2 = \begin{pmatrix}
0 \\
0 \\
1 \\
-1
\end{pmatrix}.
\]

(15)

Therefore, the kinematic model of $\tilde{\Sigma}_{RS}^B$ is:

\[
\dot{\tilde{\gamma}} = g_1 u_1 + g_2 v_1,
\]

where $u_1 \in \{-1, 1\}$ and

\[
\max \left\{-1, \frac{4u_1 \sin \gamma_1}{D} - 1\right\} \leq v_1 \leq \min \left\{\frac{4u_1 \sin \gamma_1}{D} + 1, 1\right\}.
\]

(17)

To satisfy (17) we have $\frac{4u_1 \sin \gamma_1}{D} - 1 \leq 1$ and $1 + \frac{4u_1 \sin \gamma_1}{D} \geq -1$, hence

\[
|\sin \gamma_1| \leq \frac{D}{2}.
\]

(18)

It is now important to explicit the dependence of the results with respect to $R_{\text{min}}$. Indeed, (18) would be $|\sin \gamma_1| \leq \frac{D}{2R_{\text{min}}}$. If $D > 2R_{\text{min}}$, $\gamma_1 \in S^1$. On the other hand, if $D \leq 2R_{\text{min}}$, $\gamma_1 \in \Gamma_I \cup \Gamma_{II} \cup \Gamma_s$, where

\[
\Gamma_I = [\arcsin(\frac{D}{2}), \arcsin(\frac{D}{2})], \\
\Gamma_{II} = [-\arcsin(\frac{D}{2}) + \pi, \arcsin(\frac{D}{2}) + \pi], \\
\Gamma_s = \{\gamma_1| |\sin \gamma_1| = \frac{D}{2}\}.
\]

(19)

For example, for $D = R_{\text{min}}$ feasible configurations are represented in fig. 10 and admissible controls $(\Gamma_I = [-\frac{\pi}{6}, \frac{\pi}{6}], \Gamma_{II} = [-\frac{\pi}{6}, \frac{\pi}{6} + \frac{\pi}{6}], \Gamma_s = \{\gamma_1| |\sin \gamma_1| = \frac{D}{2}\})$ are represented in fig. 11.

We denote with $\mathcal{M}_{RS}^{B^+} = \mathbb{R}^2 \times S^1 \times S^1$ the configuration space when $D > 2R_{\text{min}}$, and with $\mathcal{M}_{RS}^{B^-} = \mathbb{R}^2 \times S^1 \times \Gamma_I \cup \Gamma_{II}$ and $\mathcal{M}_{RS}^{B^0} = \mathbb{R}^2 \times S^1 \times \Gamma_s$ the configuration space when $D \leq 2R_{\text{min}}$. Notice that $\mathcal{M}_{RS}^{B^0}$ consists of singular configurations.

Lemma 2: For $\mathcal{M}_{RS}^{B^+}$, ARC holds at any $\tilde{\gamma} \in \mathcal{M}_{RS}^{B^+}$ if $D > 2R_{\text{min}}$ and $\tilde{\gamma} \in \mathcal{M}_{RS}^{B^-}$ if $D \leq 2R_{\text{min}}$. But ARC fails at $\tilde{\gamma} \in \mathcal{M}_{RS}^{B^0}$.

Proof: We start applying remark 1 for $D \leq 2R_{\text{min}}$ and $\tilde{\gamma} \in \mathcal{M}_{RS}^{B^0}$. In this case $\sin \gamma_1 = \frac{D}{2R_{\text{min}}}$. The only two admissible controls are either $(u_1, v_1) = (-u_1^2, v_1^2) = (1, 1)$ or $(u_1, v_1) = (-u_1^2, v_1^2) = (-1, 1)$, see fig.12. Notice that
The accessibility rank condition holds and hence the thesis.

**Remark 2.** Then system can evolves in $\Sigma_{RS}$ for $D = 2R_{min}$ and centered at $(x_1, y_1, \theta_1, 0)$ that evolves in $\mathcal{M}^B_{RS}$ always exists for Lemma 3, see fig.15. The system then evolves in $\mathcal{M}^A_{RS}$ (as $\Sigma_{RS}$), and can reach $q^2 = (x_1, y_1, \theta_1, 0)$ for some $(x_1, y_1, \theta_1)$ for Remark 2. Then system can evolves in $\mathcal{M}^B_I$ to achieve any $q^f \in \mathcal{M}^B_I$ for lemma 3. There exists an equivalent control law that steers the system from $q^i \in \mathcal{M}^I_I$ to $q^f \in \mathcal{M}^I_I$.

We now prove that for $D < 2R_{min}$, the system can be steered between any two configurations in $\mathcal{M}^B_{RS}$ crossing $\mathcal{M}^B_{RS}$. Without loss of generality let $q^0 \in \mathcal{M}^B_{RS} \cap \mathcal{M}^B_{RS}$, a trajectory from $q^0$ to $q^1 = (x_1, y_1, \theta_1, \pi)$ for some $(x_1, y_1, \theta_1)$ that evolves in $\mathcal{M}^B_{RS}$ always exists for Lemma 3, see fig.16. The system then evolves in $\mathcal{M}^B_{RS}$ (as $\Sigma_{RS}$), and can reach $q^2 = (x_1, y_1, \theta_1, 0)$ for some $(x_1, y_1, \theta_1)$ for Remark 2. Then system can evolves in $\mathcal{M}^I_I$ to achieve any $q^f \in \mathcal{M}^I_I$ for lemma 3. There exists an equivalent control law that steers the system from $q^i \in \mathcal{M}^I_I$ to $q^f \in \mathcal{M}^I_I$.

**Theorem 7:** $\Sigma_{RS}$ is controllable on $\mathcal{M}^A_{RS} = \mathcal{M}^A_{RS} \cup \mathcal{M}^B_{RS}$.

**Proof:** Corollary 1 states that $\Sigma_{RS}$ is controllable for both $\phi = \phi^*$ and $\phi \neq \phi^*$.

**VI. CONTROLLABILITY FOR DUBINS**

For Dubins vehicles, $u_i = 1$, hence from (8), we have:

$$\cos \gamma_1 = \cos \gamma_2.$$  \hspace{1cm} (21)

Thus we have two possible angular relationships between two vehicles

$$a : \gamma_1 = \gamma_2; \ b : \gamma_1 = -\gamma_2.$$  \hspace{1cm} (22)

Angular relationships and their intersection cases $a \land b : \gamma_1 = \gamma_2 = k\pi, k = 0, 1, \alpha$ are reported in fig.17.

Using the same reasoning used for $\Sigma_{RS}$, when $a : \gamma_1 = \gamma_2$ ($v_1 = v_2$), the kinematic model of $\Sigma_{Dubins}$ is
the reachable configurations is a limit circle.

The controllability of \( \Sigma \) requires similar reasoning as the controllability of \( \Sigma_{RS} \). However, it is much more challenging to prove that \( \Sigma \) is a weakly reversible system, details of the proof can be found in [17]. For space limitations, we only report the controllability results for \( \Sigma \).

Let the distances \( D = R_{min} \).

Theorem 8: [17] \( \Sigma_{Dubins} \) is controllable on the configuration space \( M_{Dubins} = M^A_{Dubins} \cup M^B_{Dubins} \).

When \( D \leq R_{min} \), if \( q \in M^B_{Dubins} = \mathbb{R}^2 \times S^1 \times \Gamma_{Dubins} \), the reachable configurations is a limit circle.

VII. CONTROLLABILITY FOR \( n \) VEHICLES

This section gives a direct extension of above controllability results for pairs of vehicles to \( n \) identical vehicles both for a star formation with a leader and for a chain formation.

A. Controllability for \( n \) vehicles with star formation

Given \( n \) vehicles \( V_i \), \( i = 1, \cdots, n \), let \( V_1 \) be a leader. Assume the distances \( D_i, i = 2, \cdots, n \) between \( V_1 \) and \( V_i \) are different such that no collision between vehicle occurs. Let \( \gamma_{1,i} (\gamma_{i,1}), i = 2, \cdots, n \) denote the angle from the heading direction of \( V_1 (V_i) \) to the distance direction from \( V_1 \) to \( V_i \), see fig. 19. Such system is denoted by \( \Sigma^n_{a} \).

Let \( \hat{q} = (x_1, y_1, \theta_1, \gamma_{1,2}, \gamma_{1,3}, \cdots, \gamma_{1,n}, \gamma_{n,1}) \) be the configuration of \( \Sigma^n_{a} \). If vehicles are all DDV, Car or CRS types, the configuration spaces can be written as \( M = \mathbb{R}^2 \times S^1 \times \cdots \times S^1 \). From Theorems 4, 5 and 6 corresponding \( \Sigma^n_{a} \) are completely controllable.

For RS and Dubins vehicles, since for \( D_i \leq 2R_{min} \) the admissible control \( v_1 \) does not exist for all possible configurations, we assume that \( D_i > 2R_{min} \) for all \( i = 1, \cdots, n \). For RS vehicle, define \( S^A = S^1 \) with angular relation \( a_1 : \gamma_{1,i} = \gamma_{i,1} \) and \( a_2 : \gamma_{i,1} = \gamma_{i,1} - \pi \); \( S^B = S^1 \) with angular relation \( b_1 : \gamma_{1,i} = -\gamma_{i,1} \) and \( b_2 : \gamma_{i,1} = -\gamma_{i,1} - \pi \). For Dubins vehi-
cles, define $S^A = S^1$ with angular relation $\alpha : \gamma_{i,i+1} = \gamma_{i,1}$ and $S^B = S^1$ with angular relation $\beta : \gamma_{i,i+1} = -\gamma_{i,1}$. Then we can write $M = \mathbb{R}^2 \times S^1 \times S^1 \cup S^B \times \cdots \times S^1 \times S^A \cup S^B$. From Theorems 7 and 8 corresponding $\Sigma_i^n$ are completely controllable.

B. Controllability for $n$ vehicles with chain formation

This part gives another extension of controllability results for chain formations consisting of $n$ vehicles $V_i$, $i = 1, \ldots, n$. Assume the distances $D_i$, $i = 1, \ldots, n-1$ between $V_i$ to $V_{i+1}$ are specified such that no collision between vehicle occurs. Let $\gamma_{i,i+1} : \gamma_{i,i+1}, i = 1, \ldots, n-1$ denote the angle from the heading direction of $V_i$ ($V_{i+1}$) to the distance direction from $V_i$ to $V_{i+1}$, see fig.20. Such system is denoted by $\Sigma_c$.

Let $\bar{q} = (x_1, y_1, \theta_1, \gamma_{1,2}, \gamma_{2,3}, \cdots, \gamma_{n-1, n}, \gamma_{n,n-1})$ be the configuration of $\Sigma_c^n$. If vehicles are all DDV, Car or CRS types, the configuration spaces can be written as $M = \mathbb{R}^2 \times S^1 \times \cdots \times S^1$. From Theorems 4, 5 and 6 corresponding $\Sigma_i^n$ are completely controllable.

For RS and Dubins vehicles, we assume that all distance $D_i > 2R_{min}$. For RS vehicle, define $S^A = S^1$ with angular relation $a_1 : \gamma_{i,i+1} = \gamma_{i,1}$ and $a_2 : \gamma_{i,i+1} = -\gamma_{i,1} + \pi$; $S^B = S^1$ with angular relation $b_1 : \gamma_{i,i+1} = -\gamma_{i,1}$ and $b_2 : \gamma_{i,i+1} = -\gamma_{i,1} + \pi$. For Dubins vehicles, define $S^A = S^1$ with angular relation $a : \gamma_{i,i+1} = \gamma_{i+1,i}$ and $S^B = S^1$ with angular relation $b : \gamma_{i,i+1} = -\gamma_{i,1}$. Then we can write $M = \mathbb{R}^2 \times S^1 \times S^1 \times S^A \cup S^B \times \cdots \times S^1 \times S^A \cup S^B$. From Theorems 7 and 8 corresponding $\Sigma_i^n$ are completely controllable.

VIII. CONCLUSIONS

This paper has provided controllability results for pairs of identical vehicles (Dubins, Reeds-Shepp, differential drive, car-like and convexified Reeds-Shepp) that move maintaining a constant distance. Known theorems of controllability have been extended to solve the controllability problem for special affine control systems whose admissible control domains depend on their configurations. Furthermore, a practical condition has been provided to apply the proposed theorem for studied systems.

As a result, for differential drive, car-like and convexified Reeds-Shepp vehicles complete controllability has been proved. The same does not hold for pairs of Dubins or Reeds-Shepp vehicles, and a description of the reachable sets in these cases has been provided. Limit circles for particular configurations have been proved to exist in case of small distance to be maintained.

Finally, controllability results have been presented, as a direct extension of pairs, for $n$ identical vehicles with star formations and chain formations. The optimal control for larger groups of robots are under study.

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REFERENCES